QUANTITATIVE METHODS for ECONOMIC APPLICATIONS
MATHEMATICS for ECONOMIC APPLICATIONS
TASK 28/6/2021

I M 1) The polynomial equation $x^3 - i\,x^2 - x + i = 0$ has two real solutions and one imaginary solution. Calculate the cubic roots of the product of the three solutions.

For $x = i$ we get $P(i) = i^3 - i\,i^2 - i + i = -i + i - i + i = 0$.
So $x = i$ is the imaginary solution. By Ruffini's rule we get:

$$\begin{array}{c|ccc|c} & 1 & -i & -1 & i \\ i & & i & 0 & -i \\ \hline & 1 & 0 & -1 & 0 \end{array}$$

and so $x^3 - i\,x^2 - x + i = (x - i)\left(x^2 - 1\right) = 0$ and we get the three solutions $x_1 = i$, $x_2 = 1$, $x_3 = -1$ and $x_1 \cdot x_2 \cdot x_3 = -i = \cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi$.

So $\sqrt[3]{-i} = \cos\left(\frac{1}{2}\pi + k\,\frac{2\pi}{3}\right) + i\sin\left(\frac{1}{2}\pi + k\,\frac{2\pi}{3}\right), 0 \le k \le 2$.

For $k = 0$: $z_0 = \cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi = i$;

for $k = 1$: $z_1 = \cos\frac{7}{6}\pi + i\sin\frac{7}{6}\pi = -\frac{\sqrt{3}}{2} - i\,\frac{1}{2}$;

for $k = 2$: $z_2 = \cos\frac{11}{6}\pi + i\sin\frac{11}{6}\pi = \frac{\sqrt{3}}{2} - i\,\frac{1}{2}$.

I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} 2 & 3 & -1 \\ 2 & 1 & 2 \\ 2 & 5 & k \end{Vmatrix}$, the linear map $\mathbb{R}^3 \to \mathbb{R}^3 : \mathbb{A} \cdot \mathbb{X} = \mathbb{Y}$, and the vector $\mathbb{Y} = \begin{Vmatrix} 1 \\ 2 \\ m \end{Vmatrix}$, determine for which values of the parameters k and m the vector $\mathbb{Y}$ belongs to the Image of the map.

This problem corresponds perfectly to see when the linear system $\mathbb{A} \cdot \mathbb{X} = \mathbb{Y}$ admits solutions, and so we use Rouchè-Capelli Theorem and we calculate the rank of the matrix and the rank of the augmented matrix :

$(\mathbb{A}|\mathbb{Y}) = \begin{Vmatrix} 2 & 3 & -1 & | & 1 \\ 2 & 1 & 2 & | & 2 \\ 2 & 5 & k & | & m \end{Vmatrix}$. By elementary operations on the rows:

$(R_2 \leftarrow R_2 - R_1)$, $(R_3 \leftarrow R_3 - R_1)$, and then $(R_3 \leftarrow R_3 + R_2)$ we get :

$$\begin{Vmatrix} 2 & 3 & -1 & | & 1 \\ 2 & 1 & 2 & | & 2 \\ 2 & 5 & k & | & m \end{Vmatrix} \to \begin{Vmatrix} 2 & 3 & -1 & | & 1 \\ 0 & -2 & 3 & | & 1 \\ 0 & 2 & k+1 & | & m-1 \end{Vmatrix} \to \begin{Vmatrix} 2 & 3 & -1 & | & 1 \\ 0 & -2 & 3 & | & 1 \\ 0 & 0 & k+4 & | & m \end{Vmatrix}$$

and so we see that:
$-$ if $k \neq -4 \Rightarrow \text{Rank}(\mathbb{A}) = 3 = \text{Rank}(\mathbb{A}|\mathbb{Y})$: the system has one and only one solution : the vector $\mathbb{Y}$ is image of a unique vector $\mathbb{X}$;
$-$ if $k = -4$ and $m = 0 \Rightarrow \text{Rank}(\mathbb{A}) = 2 = \text{Rank}(\mathbb{A}|\mathbb{Y})$: the system has ∞^1 solutions : the vector $\mathbb{Y}$ is an image of infinite vectors $\mathbb{X}$;
$-$ if $k = -4$ and $m \neq 0 \Rightarrow \text{Rank}(\mathbb{A}) = 2 < \text{Rank}(\mathbb{A}|\mathbb{Y}) = 3$: the system has no solutions. the vector $\mathbb{Y}$ does not belong to the Image of the map.

I M 3) Given the matrix $\mathbb{A} = \left\|\begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}\right\|$, determine an orthogonal matrix that diagonalizes it.

The matrix is symmetric and therefore diagonalizable by an orthogonal matrix.
Let's determine the eigenvalues of the matrix:

From $|\mathbb{A} - \lambda\,\mathbb{I}| = \begin{vmatrix} -\lambda & 0 & 1 & 1 \\ 0 & -\lambda & 1 & 1 \\ 1 & 1 & -\lambda & 0 \\ 1 & 1 & 0 & -\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & \lambda & 0 & 0 \\ 0 & -\lambda & 1 & 1 \\ 1 & 1 & -\lambda & 0 \\ 0 & 0 & \lambda & -\lambda \end{vmatrix} =$

by elementary operations on the rows $(R_1 \leftarrow R_1 - R_2)$, $(R_4 \leftarrow R_4 - R_3)$

$$= (-\lambda) \cdot \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 0 & \lambda & -\lambda \end{vmatrix} - \lambda \cdot \begin{vmatrix} 0 & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} =$$

by elementary operations on the columns $(C_2 \leftarrow C_2 + C_3)$ and $(C_2 \leftarrow C_2 - C_3)$:

$$= (-\lambda) \cdot \begin{vmatrix} -\lambda & 2 & 1 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} - \lambda \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & -\lambda & 0 \\ 1 & \lambda & -\lambda \end{vmatrix} =$$

$$= (-\lambda) \cdot (-\lambda) \cdot \begin{vmatrix} -\lambda & 2 \\ 1 & -\lambda \end{vmatrix} - \lambda \cdot 1 \cdot \begin{vmatrix} 1 & -\lambda \\ 1 & \lambda \end{vmatrix} =$$

$$= \lambda^2 \cdot \left(\lambda^2 - 2\right) - \lambda \cdot (\lambda + \lambda) = \lambda^2 \cdot \left(\lambda^2 - 2 - 2\right) = \lambda^2 \cdot \left(\lambda^2 - 4\right) = 0\,.$$

So the eigenvalues are $\lambda_1 = \lambda_2 = 0, \lambda_3 = 2, \lambda_4 = -2$.
To get the eigenvectors corresponding to $\lambda_1 = \lambda_2 = 0$ we solve the homogeneous system :

$$\|\mathbb{A} - 0 \cdot \mathbb{I}\| \cdot \mathbb{X} = \mathbb{A} \cdot \mathbb{X} = \mathbb{O} \Rightarrow \left\|\begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \\ z \\ w \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right\| \Rightarrow \begin{cases} z + w = 0 \\ x + y = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} y = -x \\ w = -z \end{cases} \Rightarrow (x, -x, z, -z)$$ the eigenvectors corresponding to $\lambda = 0$.

To get two orthogonal eigenvectors we choose $x = 1, z = 0$ and then $x = 0, z = 1$: $\mathbb{X}_1 = (1, -1, 0, 0)$ and $\mathbb{X}_2 = (0, 0, 1, -1)$.
To get the eigenvectors corresponding to $\lambda_3 = 2$ we solve the homogeneous system :

$$\|\mathbb{A} - 2 \cdot \mathbb{I}\| \cdot \mathbb{X} = \mathbb{O} \Rightarrow \left\|\begin{matrix} -2 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & -2 \end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \\ z \\ w \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right\| \Rightarrow$$

$$\Rightarrow \left\| \begin{matrix} -2 & 0 & 1 & 1 \\ 2 & -2 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & -2 \end{matrix} \right\| \cdot \left\| \begin{matrix} x \\ y \\ z \\ w \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\| \Rightarrow \begin{cases} -2x+z+w=0 \\ 2x-2y=0 \\ x+y-2z=0 \\ 2z-2w=0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} y=x \\ w=z \\ -2x+2z=0 \\ 2x-2z=0 \end{cases} \Rightarrow \begin{cases} y=x \\ z=x \\ w=x \end{cases} \Rightarrow (x,x,x,x) \text{ the eigenvectors corresponding to } \lambda=2$$

and so, for $x=1$ we get $\mathbb{X}_3=(1,1,1,1)$.
To get the eigenvectors corresponding to $\lambda_4=-2$ we solve the homogeneous system :

$$\|\mathbb{A}+2\cdot\mathbb{I}\|\cdot\mathbb{X}=\mathbb{O} \Rightarrow \left\| \begin{matrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{matrix} \right\| \cdot \left\| \begin{matrix} x \\ y \\ z \\ w \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\| \Rightarrow$$

$$\Rightarrow \left\| \begin{matrix} 2 & 0 & 1 & 1 \\ -2 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & -2 & 2 \end{matrix} \right\| \cdot \left\| \begin{matrix} x \\ y \\ z \\ w \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\| \Rightarrow \begin{cases} 2x+z+w=0 \\ -2x+2y=0 \\ x+y+2z=0 \\ -2z+2w=0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} y=x \\ w=z \\ 2x+2z=0 \\ 2x+2z=0 \end{cases} \Rightarrow \begin{cases} y=x \\ z=-x \\ w=z \end{cases} \Rightarrow (x,x,-x,-x) \text{ the eigenvectors corresponding to}$$

$\lambda=-2$ and so, for $x=1$ we get $\mathbb{X}_4=(1,1,-1,-1)$.
To obtain an orthogonal matrix, we finally calculate the unit vectors of the four eigenvectors found that we use as columns of the orthogonal matrix and we will have:

$$\mathbb{U} = \left\| \begin{matrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{matrix} \right\|.$$

I M 4) Find the coordinates of the vector $\mathbb{X}$ in the basis $\mathbb{A}=\{(1,1,1),(1,1,0),(1,0,1)\}$ if it has coordinates $(2,1,2)$ in the basis $\mathbb{B}=\{(1,0,0),(1,1,0),(1,1,1)\}$

Since $\mathbb{X}$ has coordinates $(2,1,2)$ in the basis $\mathbb{B}=\{(1,0,0),(1,1,0),(1,1,1)\}$ we get:

$$\mathbb{X} = \left\| \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \right\| \cdot \left\| \begin{matrix} 2 \\ 1 \\ 2 \end{matrix} \right\| = \left\| \begin{matrix} 2+1+2 \\ 0+1+2 \\ 0+0+2 \end{matrix} \right\| = \left\| \begin{matrix} 5 \\ 3 \\ 2 \end{matrix} \right\|;$$

to find the coordinates of the vector $\mathbb{X}$ in the basis $\mathbb{A}=\{(1,1,1),(1,1,0),(1,0,1)\}$ we must solve the system:

$$\left\| \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix} \right\| \cdot \left\| \begin{matrix} x \\ y \\ z \end{matrix} \right\| = \left\| \begin{matrix} 5 \\ 3 \\ 2 \end{matrix} \right\| \Rightarrow \begin{cases} x+y+z=5 \\ x+y=3 \\ x+z=2 \end{cases} \Rightarrow \begin{cases} x+3-x+2-x=5 \\ y=3-x \\ z=2-x \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x=0 \\ y=3 \\ z=2 \end{cases} \text{ are the coordinates of vector } \mathbb{X} \text{ in the basis } \mathbb{A}=\{(1,1,1),(1,1,0),(1,0,1)\}.$$

II M 1) Given the system $\begin{cases} f(x,y,z) = e^{x-y} + e^{x-z} - 2\,e^{y-z} = 0 \\ g(x,y,z) = x^3 + y^3 + z^2 - 3xyz = 0 \end{cases}$ and $\mathrm{P} = (1,1,1)$, point satisfying it, an implicit function is determined $x \to (y,z)$; of this function calculate the derivatives in the considered point.

The functions $f(x,y,z)$ and $g(x,y,z)$ are differentiable $\forall\,(x,y.z) \in \mathbb{R}^3$.
Furthermore $\begin{cases} f(1,1,1) = 1 + 1 - 2 = 0 \\ g(1,1,1) = 1 + 1 + 1 - 3 = 0 \end{cases}$. From:

$$\mathbb{J}(x,y,z) = \frac{\partial(f,g)}{\partial(x,y,z)} = \begin{Vmatrix} e^{x-y} + e^{x-z} & -\,e^{x-y} - 2\,e^{y-z} & -\,e^{x-z} + 2\,e^{y-z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 2z - 3xy \end{Vmatrix}$$

we get $\dfrac{\partial(f,g)}{\partial(x,y,z)}(1,1,1) = \begin{Vmatrix} 2 & -3 & 1 \\ 0 & 0 & -1 \end{Vmatrix}$.

Since $\begin{vmatrix} -3 & 1 \\ 0 & -1 \end{vmatrix} = 3 \neq 0$ it is possible to define an implicit function $x \to (y,z)$ whose derivatives are:

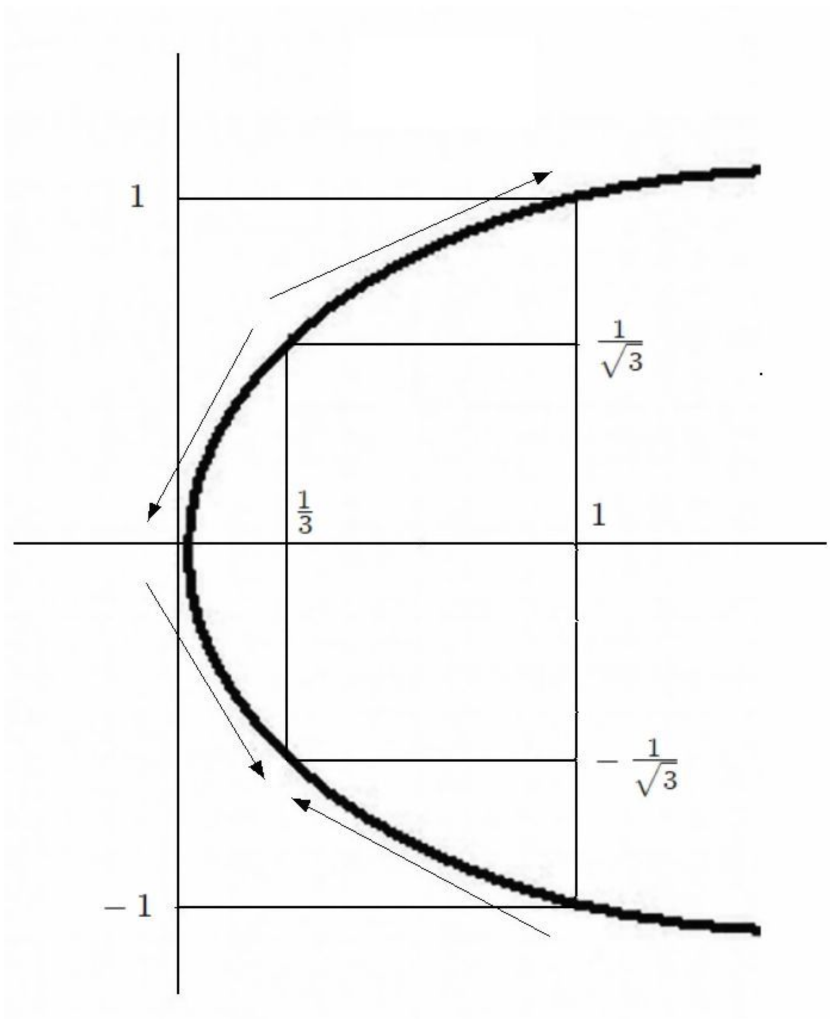
$$\frac{\mathrm{d}y}{\mathrm{d}x}(1) = -\frac{\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} -3 & 1 \\ 0 & -1 \end{vmatrix}} = -\frac{-2}{3} = \frac{2}{3} \text{ and } \frac{\mathrm{d}z}{\mathrm{d}x}(1) = -\frac{\begin{vmatrix} -3 & 2 \\ 0 & 0 \end{vmatrix}}{\begin{vmatrix} -3 & 1 \\ 0 & -1 \end{vmatrix}} = 0\,.$$

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = y\,(x-1) \\ \text{u.c.: } \begin{cases} x \le 1 \\ x \ge y^2 \end{cases} \end{cases}$.

The objective function of the problem is a continuous function, the constraints define a feasible region which is a closed and bounded (compact) set, and so we can apply Weierstrass Theorem. Surely the function admits maximum value and minimum value.
If we study the sign of the function we get:

$$f(x,y) = y\,(x-1) > 0 \Rightarrow \begin{cases} x > 1 \\ y > 0 \end{cases} \cup \begin{cases} x < 1 \\ y < 0 \end{cases} \Rightarrow y\,(x-1) > 0 \text{ for } \begin{cases} x < 1 \\ y < 0 \end{cases}.$$

In all the points $x = 1$ the objective function is equal to zero.



To solve the problem we use the Kuhn-Tucker conditions.

We write the problem as $\begin{cases} \text{Max/min } f(x,y) = y\,(x-1) \\ \text{u.c.: } \begin{cases} y^2 - x \le 0 \\ x - 1 \le 0 \end{cases} \end{cases}$.

We form the Lagrangian function:

$\Lambda(x,y,\lambda) = xy - y - \lambda_1\big(y^2 - x\big) - \lambda_2(x-1)\,.$

By applying the first order conditions we have:

1) case $\lambda_1 = 0, \lambda_2 = 0$:

$\begin{cases} \Lambda'_x = y = 0 \\ \Lambda'_y = x - 1 = 0 \\ y^2 - x \le 0 \\ x \le 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ 0 - 1 \le 0 \\ 1 \le 1 \end{cases}$. But $f(1,0) = 0$ and since in every neighborhood of the point $(1,0)$ there are both points where the function is positive and points where it is negative, the point $(1,0)$ is a saddle point.

2) case $\lambda_1 \ne 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = y + \lambda_1 = 0 \\ \Lambda'_y = x - 1 - 2\lambda_1 y = 0 \\ x = y^2 \\ x \le 1 \end{cases} \Rightarrow \begin{cases} y = -\lambda_1 \\ x - 1 + 2\lambda_1^2 = 0 \\ x = y^2 \\ x \le 1 \end{cases} \Rightarrow \begin{cases} x = 1 - 2\lambda_1^2 \\ y = -\lambda_1 \\ 1 - 2\lambda_1^2 = \lambda_1^2 \\ x \le 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = 1 - 2\lambda_1^2 \\ y = -\lambda_1 \\ 3\lambda_1^2 = 1 \\ x \le 1 \end{cases} \Rightarrow \begin{cases} x = 1 - 2\lambda_1^2 \\ y = -\lambda_1 \\ \lambda_1^2 = \frac{1}{3} \\ x \le 1 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{3} \\ y = -\frac{1}{\sqrt{3}} \\ \lambda_1 = \frac{1}{\sqrt{3}} \\ \frac{1}{3} \le 1 \end{cases} \cup \begin{cases} x = \frac{1}{3} \\ y = \frac{1}{\sqrt{3}} \\ \lambda_1 = -\frac{1}{\sqrt{3}} \\ \frac{1}{3} \le 1 \end{cases}.$$

The point $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$, since $\lambda_1 = \frac{1}{\sqrt{3}} > 0$, may be a maximum point, the point $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$, since $\lambda_1 = -\frac{1}{\sqrt{3}} < 0$, may be a minimum point.

3) case $\lambda_1 = 0, \lambda_2 \neq 0$:

$$\begin{cases} \Lambda'_x = y - \lambda_2 = 0 \\ \Lambda'_y = x - 1 = 0 \\ x = 1 \\ y^2 \le x \end{cases} \Rightarrow \begin{cases} x = 1 \\ x = 1 \\ y = \lambda_2 \\ \lambda_2^2 \le 1 \end{cases}$$

: we know that for all points where $x = 1$ the function is equal to 0. So in the first quadrant, where the function is negative, these are maximum points while in the fourth quadrant, where the function is positive, these are minimum points.
The point $(1, 0)$, as already seen, is a saddle point.

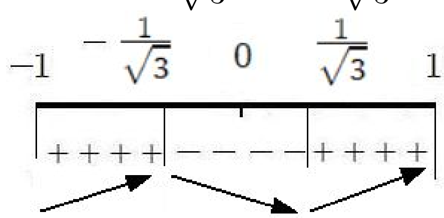
4) case $\lambda_1 \neq 0, \lambda_2 \neq 0$:

$$\begin{cases} \Lambda'_x = y + \lambda_1 - \lambda_2 = 0 \\ \Lambda'_y = x - 1 - 2\lambda_1 y = 0 \\ y^2 = 1 \\ x = 1 \end{cases} \Rightarrow \begin{cases} \Lambda'_x = y + \lambda_1 - \lambda_2 = 0 \\ \Lambda'_y = x - 1 - 2\lambda_1 y = 0 \\ y = \pm 1 \\ x = 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = 1 \\ y = 1 \\ 1 + \lambda_1 - \lambda_2 = 0 \\ 2\lambda_1 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \\ \lambda_1 = 0 \\ \lambda_2 = 1 \end{cases} \cup \begin{cases} x = 1 \\ y = -1 \\ -1 + \lambda_1 - \lambda_2 = 0 \\ 2\lambda_1 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -1 \\ \lambda_1 = 0 \\ \lambda_2 = -1 \end{cases}.$$

As we have already seen, the point $(1, 1)$ is reported as a maximum point while the point $(1, -1)$ is reported as a minimum point.

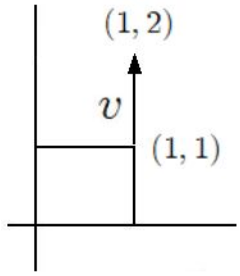
In all the points $x = 1$ the objective function is equal to zero.
In the points $x = y^2$ the objective function becomes $f\left(y^2, y\right) = y\left(y^2 - 1\right) = y^3 - y$ and so:
$f'(y) = 3y^2 - 1 \Rightarrow f'(y) \ge 0$ for $y \le -\frac{1}{\sqrt{3}}$ and for $\frac{1}{\sqrt{3}} \le y$.



We get the same results found with the case $\lambda_1 \neq 0, \lambda_2 = 0$.

The point $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ with $f\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$ is the maximum point, the point $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ with $f\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$ is the minimum point.
All the points $x = 1; 0 < y \le 1$ are maximum points, all the points $x = 1; -1 \le y < 0$ are minimum points.

II M 3) Given $f(x,y) = 2x\,y - x^2$, calculate $\mathcal{D}_v f(1,1)$, where v represents the direction from $(1,1)$ to $P_0 = (1,2)$.



The function $f(x,y) = 2x\,y - x^2$ is differentiable and so $\mathcal{D}_v f(1,1) = \nabla f(1,1) \cdot v$.
From $\nabla f(x,y) = (2y - 2x, 2x)$ we get $\nabla f(1,1) = (0,2)$, since $v = (0,1)$ it is $\|v\| = 1$ and finally $\mathcal{D}_v f(1,1) = \nabla f(1,1) \cdot v = (0,2) \cdot (0,1) = 2$.

II M 4) Determine, on varying the parameter k, the nature of the stationary points of the function $f(x,y) = x^3 - 3\,kxy + y^2$.

We apply first order conditions:

$$\nabla f(x,y) = \mathbb{O} \Rightarrow \begin{cases} f'_x = 3x^2 - 3ky = 0 \\ f'_y = 2y - 3\,kx = 0 \end{cases} \Rightarrow \begin{cases} 3(x^2 - ky) = 0 \\ y = \frac{3}{2}\,kx \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 3\left(x^2 - \frac{3}{2}\,k^2 x\right) = 3x\left(x - \frac{3}{2}\,k^2\right) = 0 \\ y = \frac{3}{2}\,kx \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \cup \begin{cases} x = \frac{3}{2}\,k^2 \\ y = \frac{9}{4}\,k^3 \end{cases}$$

and so we get two possible solutions: $(0,0)$ and $\left(\frac{3}{2}\,k^2, \frac{9}{4}\,k^3\right)$.
For the second order conditions we construct the Hessian matrix:

$$\mathbb{H}(x,y) = \begin{Vmatrix} 6x & -3k \\ -3k & 2 \end{Vmatrix}.$$

Since $\mathbb{H}(0,0) = \begin{Vmatrix} 0 & -3k \\ -3k & 2 \end{Vmatrix}$ it is $|\mathbb{H}_2| = -9k^2 < 0$ if $k \ne 0$ and so $(0,0)$ is a saddle point; if $k = 0$ we get $f(x,y) = x^3 + y^2 \Rightarrow f(0,0) = 0$ and it is easy to see how in every neighborhood of $(0,0)$ the function assumes both positive and negative values, and therefore again $(0,0)$ is a saddle point.

Since $\mathbb{H}\left(\frac{3}{2}\,k^2, \frac{9}{4}\,k^3\right) = \begin{Vmatrix} 9\,k^2 & -3k \\ -3k & 2 \end{Vmatrix} \Rightarrow \begin{cases} |\mathbb{H}_1| = 9\,k^2 > 0; |\mathbb{H}_1| = 2 > 0 \\ |\mathbb{H}_2| = 9\,k^2 > 0 \text{ if } k \ne 0 \end{cases}$ and so the point $\left(\frac{3}{2}\,k^2, \frac{9}{4}\,k^3\right)$ is a minimum point; the case $k = 0$ has just been studied.