QUANTITATIVE METHODS for ECONOMIC APPLICATIONS
MATHEMATICS for ECONOMIC APPLICATIONS
TASK 1/9/2021

I M 1) Calculate the square roots of the number $z = \dfrac{1}{1+i}$.

Since $1 + i = \sqrt{2}\left(\dfrac{1}{\sqrt{2}} + i\,\dfrac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$ we get:

$z = \dfrac{1}{1+i} = \dfrac{1}{\sqrt{2}}\left(\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right) = \dfrac{1}{\sqrt{2}}\left(\cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4}\right)$.

Or even $z = \dfrac{1}{1+i} = \dfrac{1-i}{(1+i)(1-i)} = \dfrac{\sqrt{2}}{2}\left(\cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4}\right)$. Then:

$\sqrt{z} = \dfrac{1}{\sqrt[4]{2}}\left(\cos\left(\dfrac{7\pi}{8} + k\pi\right) + i\sin\left(\dfrac{7\pi}{8} + k\pi\right)\right); 0 \le k \le 1$.

For $k = 0 : z_0 = \dfrac{1}{\sqrt[4]{2}}\left(\cos\dfrac{7\pi}{8} + i\sin\dfrac{7\pi}{8}\right)$;

for $k = 1 : z_1 = \dfrac{1}{\sqrt[4]{2}}\left(\cos\dfrac{15\pi}{8} + i\sin\dfrac{15\pi}{8}\right)$.

I M 2) Given the two orthogonal vectors $\mathbb{X}_1 = (1, 1, -2)$ and $\mathbb{X}_2 = (1, 1, 1)$, determine a third vector $\mathbb{X}_3$ orthogonal to both them and, with their unit vectors, form an orthogonal matrix. Finally, determine the inverse of this orthogonal matrix.

Firstly we determine a vector orthogonal to $\mathbb{X}_2 = (1, 1, 1) : (x, y, z)$.
It will be: $(1, 1, 1) \cdot (x, y, z) = x + y + z = 0 \Rightarrow z = -x - y$.
Now we determine a vector orthogonal to $\mathbb{X}_1 = (1, 1, -2) : (x, y, -x - y)$.
It will be: $(1, 1, -2) \cdot (x, y, -x - y) = x + y + 2x + 2y = 3x + 3y = 0 \Rightarrow y = -x$.
Therefore a vector orthogonal to both will be $\mathbb{X}_3 = (x, -x, 0)$.
By choosing $x = 1$ we get $\mathbb{X}_3 = (1, -1, 0)$. Their unit vectors will be :
$\overline{\mathbb{X}}_1 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$, $\overline{\mathbb{X}}_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\overline{\mathbb{X}}_3 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$. Therefore a possible orthogonal matrix will be $\mathbb{U} = \begin{Vmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{Vmatrix}$.

Since the matrix is an orthogonal one, to get its inverse it is enough to make its transpose and so we get $\mathbb{U}^{-1} = \mathbb{U}^T = \begin{Vmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -0 \end{Vmatrix}$.

I M 3) Given the matrix $\mathbb{A}=\left\|\begin{matrix} k & 0 & k \\ 0 & -1 & 0 \\ k & 0 & k \end{matrix}\right\|$, determine the values of k for which the matrix admits a multiple eigenvalue .

From $|\mathbb{A}-\lambda\,\mathbb{I}|=\begin{vmatrix} k-\lambda & 0 & k \\ 0 & -1-\lambda & 0 \\ k & 0 & k-\lambda \end{vmatrix}=\begin{vmatrix} -\lambda & 0 & k \\ 0 & -1-\lambda & 0 \\ \lambda & 0 & k-\lambda \end{vmatrix}=$
$=-\lambda\,(-1-\lambda)(k-\lambda)+\lambda\,(-k\,(-1-\lambda))=-\lambda\,(-1-\lambda)((k-\lambda)+k)=$
$=\lambda\,(1+\lambda)(2k-\lambda)=0\Rightarrow\lambda_1=0,\lambda_2=-1,\lambda_3=2k$. Hence the matrix admits multiple eigenvalues if $2k=0\Rightarrow k=0$ or if $2k=-1\Rightarrow k=-\dfrac{1}{2}$.

I M 4) Given the matrix $\mathbb{A}=\left\|\begin{matrix} 1 & m & -1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & -1 & -4 & 1 \end{matrix}\right\|$ and the vector $\mathbb{Y}=(2,6,m)$, determine for which value of m the vector $\mathbb{Y}$ belongs to the Image of the linear map generated by $\mathbb{A}$.

To see if the vector $\mathbb{Y}$ belongs to the Image of the linear map generated by the matrix $\mathbb{A}$ it is sufficient to check that the linear system having for matrix $\mathbb{A}$ and for vector of the known terms $\mathbb{Y}$ is a system that admits solutions. So we have to apply the Rouchè-Capelli Theorem.
$(\mathbb{A}|\mathbb{Y})=\left\|\begin{matrix} 1 & m & -1 & 1 & | & 2 \\ 2 & 1 & 1 & 2 & | & 6 \\ 1 & -1 & -4 & 1 & | & m \end{matrix}\right\|$. By elementary operations on the rows:
$(R_2\leftarrow R_2-2R_1)$, $(R_3\leftarrow R_3-R_1)$, and then $(R_3\leftarrow R_3+R_2)$ we get :
$(\mathbb{A}|\mathbb{Y})\rightarrow\left\|\begin{matrix} 1 & m & -1 & 1 & | & 2 \\ 0 & 1-2m & 3 & 0 & | & 2 \\ 0 & -1-m & -3 & 0 & | & m-2 \end{matrix}\right\|\rightarrow\left\|\begin{matrix} 1 & m & -1 & 1 & | & 2 \\ 0 & 1-2m & 3 & 0 & | & 2 \\ 0 & -3m & 0 & 0 & | & m \end{matrix}\right\|$.
And so, if $m=0$ we get $\text{Rank}(\mathbb{A})=2=\text{Rank}(\mathbb{A}|\mathbb{Y})$: the system has ∞^2 solutions and the vector $\mathbb{Y}$ belongs to the Image of the linear map;
If $m\neq 0$ we get $\text{Rank}(\mathbb{A})=3=\text{Rank}(\mathbb{A}|\mathbb{Y})$: the system has ∞^1 solutions and the vector $\mathbb{Y}$ belongs to the Image of the linear map.
In conclusion, the vector $\mathbb{Y}$ belongs to the Image for each value of m.

II M 1) Given $f(x,y)=x\,e^y$ and $\mathrm{P}_0=(1,0)$, find all the directions $v=(\cos\alpha,\sin\alpha)$ for which it is : $\mathcal{D}_v f(\mathrm{P}_0)=0$.

The function $f(x,y)=x\,e^y$ is differentiable and so $\mathcal{D}_v f(\mathrm{P}_0)=\nabla f(\mathrm{P}_0)\cdot v$.
From $\nabla f(x,y)=(e^y,x\,e^y)$ we get $\nabla f(1,0)=(1,1)$. Finally:
$\mathcal{D}_v f(1,0)=(1,1)\cdot(\cos\alpha,\sin\alpha)=\cos\alpha+\sin\alpha=0\Rightarrow\cos\alpha=-\sin\alpha$ and so:
$\alpha=\dfrac{3\pi}{4}$ or $\alpha=\dfrac{7\pi}{4}$.

II M 2) Solve the problem $\begin{cases}\text{Max/min } f(x,y)=2x-3y \\ \text{u.c. : } x^2+y^2\leq 4\end{cases}$.

The objective function of the problem is a continuous function, the constraint defines a feasible region, a circumference, which is a closed and bounded (compact) set, and so we can apply Weierstrass Theorem. Surely the function admits maximum value and minimum value.

To solve the problem we use the Kuhn-Tucker conditions.

We write the problem as $\begin{cases} \text{Max/min } f(x,y) = 2x - 3y \\ \text{u.c.} : x^2 + y^2 - 4 \le 0 \end{cases}$.

We form the Lagrangian function:

$\Lambda(x,y,\lambda) = 2x - 3y - \lambda\big(x^2 + y^2 - 4\big)$.

By applying the first order conditions we have:

1) case $\lambda = 0$:

$$\begin{cases} \Lambda'_x = 2 \ne 0 \\ \Lambda'_y = -3 \ne 0 \\ x^2 + y^2 \le 4 \end{cases} \Rightarrow \text{no solution}.$$

2) case $\lambda \ne 0$:

$$\begin{cases} \Lambda'_x = 2 - 2\lambda x = 2(1 - \lambda x) = 0 \\ \Lambda'_y = -3 - 2\lambda y = 0 \\ x^2 + y^2 = 4 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\lambda} \\ y = -\frac{3}{2\lambda} \\ \frac{1}{\lambda^2} + \frac{9}{4\lambda^2} = 4 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\lambda} \\ y = -\frac{3}{2\lambda} \\ \frac{1}{\lambda^2}\left(1 + \frac{9}{4}\right) = 4 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = \frac{1}{\lambda} \\ y = -\frac{3}{2\lambda} \\ \lambda^2 = \frac{13}{16} \end{cases} \Rightarrow \begin{cases} x = \dfrac{4}{\sqrt{13}} \\ y = -\dfrac{6}{\sqrt{13}} \\ \lambda = \dfrac{\sqrt{13}}{4} > 0 \end{cases} \cup \begin{cases} x = -\dfrac{4}{\sqrt{13}} \\ y = \dfrac{6}{\sqrt{13}} \\ \lambda = -\dfrac{\sqrt{13}}{4} < 0 \end{cases} .$$

Since we have only two solutions, the point $\left(\frac{4}{\sqrt{13}}, -\frac{6}{\sqrt{13}}\right)$ with $\lambda > 0$ is the maximum point, with $f\left(\frac{4}{\sqrt{13}}, -\frac{6}{\sqrt{13}}\right) = 2\sqrt{13}$ while $\left(-\frac{4}{\sqrt{13}}, \frac{6}{\sqrt{13}}\right)$ with $\lambda < 0$ is the minimum point with $f\left(-\frac{4}{\sqrt{13}}, \frac{6}{\sqrt{13}}\right) = -2\sqrt{13}$.

II M 3) Given the function $f(x,y,z) = x\,e^{y-z} + x\log(2y - x)$, calculate its gradient at the point $P_0 = (1,1,1)$.

Since:

$$\frac{\partial f}{\partial x} = e^{y-z} + \log(2y - x) + x\,\frac{1}{2y - x}(-1) = e^{y-z} + \log(2y - x) + \frac{x}{x - 2y};$$

$$\frac{\partial f}{\partial y} = x\,e^{y-z} + x\,\frac{1}{2y - x}\,2 = x\,e^{y-z} + \frac{2x}{2y - x};$$

$$\frac{\partial f}{\partial z} = x\,e^{y-z}(-1) = -x\,e^{y-z}$$

it is $\nabla f(1,1,1) = (1 + 0 - 1, 1 + 2, -1) = (0, 3, -1)$.

II M 4) Given the equation $f(x,y) = x^3 - 2y^3 - 3x + 4y = 0$, satisfied at the point $(1,1)$, verify that the hypotheses of Dini's theorem for defining an implicit function $x \to y(x)$ are satisfied and then calculate $y'(1)$.

It is $f(1,1) = 1 - 2 - 3 + 4 = 0$ and so the equation is satisfied at $(1,1)$.

The function $f(x,y)$ is differentiable $\forall\,(x,y) \in \mathbb{R}^2$. And so:
$\nabla f(x,y) = \left(3x^2 - 3,\; -6y^2 + 4\right) \Rightarrow \nabla f(1,1) = (0,\; -2)$.
Since $f'_y = \;-2 \neq 0$ the hypotheses of Dini's theorem are satisfied and we get:
$$y'(1) = -\frac{f'_x(1,1)}{f'_y(1,1)} = -\frac{0}{-2} = 0\,.$$