QUANTITATIVE METHODS for ECONOMIC APPLICATIONS
MATHEMATICS for ECONOMIC APPLICATIONS
TASK 16/9/2021

I M 1) The two complex numbers z_1 and z_2 have both modulus equal to 1, while the first has for argument $\dfrac{\pi}{4}$ and the second has as its argument $\dfrac{\pi}{3}$. From this informations, deduce the values of $\cos\dfrac{7\pi}{12}$ and of $\sin\dfrac{7\pi}{12}$.

It is $z_1 \cdot z_2 = \left(\cos\dfrac{\pi}{4} + i \operatorname{sen}\dfrac{\pi}{4}\right) \cdot \left(\cos\dfrac{\pi}{3} + i \operatorname{sen}\dfrac{\pi}{3}\right) = \left(\cos\dfrac{7\pi}{12} + i \operatorname{sen}\dfrac{7\pi}{12}\right)$

since $\dfrac{\pi}{4} + \dfrac{\pi}{3} = \dfrac{7\pi}{12}$. But:

$$z_1 \cdot z_2 = \left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) \cdot \left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right) = \left(\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\right) + i\left(\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\right)$$

and so $\cos\dfrac{7\pi}{12} + i \operatorname{sen}\dfrac{7\pi}{12} = \dfrac{\sqrt{2}-\sqrt{6}}{4} + i\,\dfrac{\sqrt{2}+\sqrt{6}}{4}$ from which:

$\cos\dfrac{7\pi}{12} = \dfrac{\sqrt{2}-\sqrt{6}}{4}$ while $\operatorname{sen}\dfrac{7\pi}{12} = \dfrac{\sqrt{2}+\sqrt{6}}{4}$.

I M 2) Given the linear map $\mathbb{R}^4 \to \mathbb{R}^3, \mathbb{Y} = \mathbb{A} \cdot \mathbb{X}$, with $\mathbb{A} = \begin{Vmatrix} 1 & -1 & 2 & -2 \\ 2 & 2 & 1 & m \\ 4 & 0 & 5 & k \end{Vmatrix}$, determine the values of m and k for which are equal the dimensions of the Kernel and the Image of this map, and for which the image of the vector $(1,1,1,1)$ is the vector $(0,4,4)$.

Since the matrix $\mathbb{A}$ has 3 rows and 4 columns, to have the same dimension for the Image and for the Kernel we need $\operatorname{Rank}(\mathbb{A}) = 2$.
By elementary operations on the rows:
$(R_2 \leftarrow R_2 - 2R_1)$, $(R_3 \leftarrow R_3 - 4R_1)$, and then $(R_3 \leftarrow R_3 - R_2)$ we get :

$$\begin{Vmatrix} 1 & -1 & 2 & -2 \\ 2 & 2 & 1 & m \\ 4 & 0 & 5 & k \end{Vmatrix} \to \begin{Vmatrix} 1 & -1 & 2 & -2 \\ 0 & 4 & -3 & m+4 \\ 0 & 4 & -3 & k+8 \end{Vmatrix} \to \begin{Vmatrix} 1 & -1 & 2 & -2 \\ 0 & 4 & -3 & m+4 \\ 0 & 0 & 0 & k-m+4 \end{Vmatrix}.$$

So, for $k - m + 4 = 0 \Rightarrow k = m - 4$ we have $\operatorname{Rank}(\mathbb{A}) = 2 = \operatorname{Dim}(\operatorname{Imm}(\mathbb{A}))$ from which then $\operatorname{Dim}(\operatorname{Ker}(\mathbb{A})) = 4 - \operatorname{Rank}(\mathbb{A}) = 4 - 2 = 2$. Finally:

$$\begin{Vmatrix} 1 & -1 & 2 & -2 \\ 2 & 2 & 1 & m \\ 4 & 0 & 5 & k \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 1 & -1 & 2 & -2 \\ 2 & 2 & 1 & m \\ 4 & 0 & 5 & m-4 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 1-1+2-2 \\ 2+2+1+m \\ 4+0+5+m-4 \end{Vmatrix} = .$$

$$= \begin{Vmatrix} 0 \\ 5+m \\ 5+m \end{Vmatrix} = \begin{Vmatrix} 0 \\ 4 \\ 4 \end{Vmatrix} \Rightarrow m = -1 \text{ and } k = -5 \text{. So } \mathbb{A} = \begin{Vmatrix} 1 & -1 & 2 & -2 \\ 2 & 2 & 1 & -1 \\ 4 & 0 & 5 & -5 \end{Vmatrix}.$$

I M 3) Given the matrix $\mathbb{A}=\left\|\begin{matrix}0 & 1 & m\\ 1 & 0 & 2\\ 1 & k & 0\end{matrix}\right\|$, determine some value of m and k for which the eigenvectors of the matrix are all orthogonal each other.

We know that symmetric matrices have all real eigenvalues and orthogonal eigenvectors. That is, from a symmetric matrix of order n we can always find n eigenvectors orthogonal to each other. So it is sufficient to set $m=1$ and $k=2$ to have a matrix with eigenvectors each other orthogonal two by two.

I M 4) Determine the eigenvalues of the matrix $\mathbb{A}=\left\|\begin{matrix}1 & 0 & 1\\ 0 & 1 & -1\\ 1 & -1 & k\end{matrix}\right\|$, knowing that the matrix has no inverse matrix.

We know that to be invertible a matrix must be non-singular, that is, its determinant must be different from zero. Since the matrix is not invertible, this means that its determinant is equal to zero, and since the determinant is equal to the product of the eigenvalues, at least one eigenvalue will be zero. Let's calculate the determinant:

$$|\mathbb{A}|=\begin{vmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 1 & -1 & k\end{vmatrix}=\begin{vmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & -1 & k-1\end{vmatrix}=1\cdot(k-1)-1=k-2=0\Rightarrow k=2\,.$$

Let's calculate the eigenvalues. From

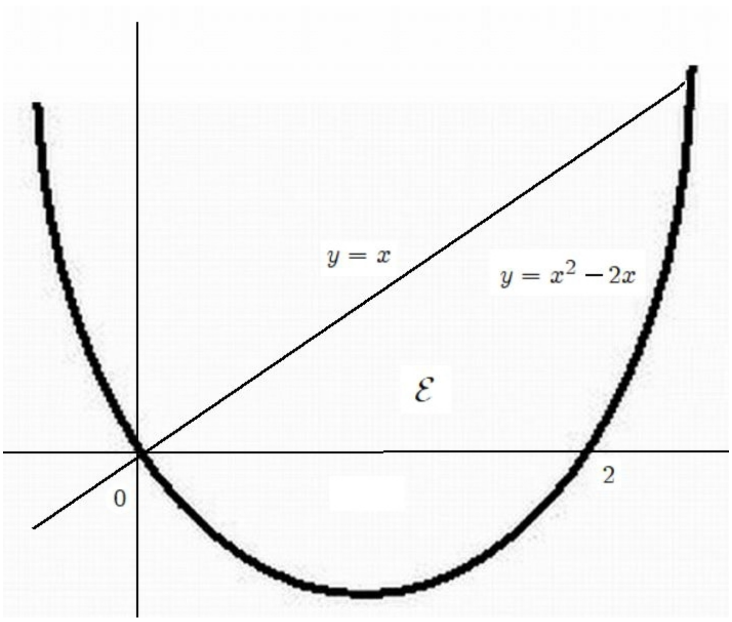
$$|\mathbb{A}-\lambda\,\mathbb{I}|=\begin{vmatrix}1-\lambda & 0 & 1\\ 0 & 1-\lambda & -1\\ 1 & -1 & 2-\lambda\end{vmatrix}=(1-\lambda)((1-\lambda)(2-\lambda)-1)+1\,(0-(1-\lambda))=$$

$$=(1-\lambda)\big(\lambda^2-3\lambda+1-1\big)=\lambda\,(1-\lambda)(\lambda-3)=0 \text{ and so } \lambda_1=0,\lambda_2=1,\lambda_3=3\,.$$

II M 1) Given the equation $f(x,y)=e^{x-y}+e^{y-x}-2\,e^{x+y-2}=0$ satisfied at the point $\mathrm{P}=(1,1)$, an implicit function $x\to y(x)$ is determined; of this function calculate the first derivative at the point $x=1$.

The function $f(x,y)=e^{x-y}+e^{y-x}-2\,e^{x+y-2}$ is differentiable $\forall\,(x,y)\in\mathbb{R}^2$.
Furthermore $f(1,1)=1+1-2\;=0$. From:
$\nabla f(x,y)=\left(e^{x-y}-e^{y-x}-2\,e^{x+y-2};\;-e^{x-y}+e^{y-x}-2\,e^{x+y-2}\right)$
we get: $\nabla f(1,1)=(1-1-2;\;-1+1-2)=(-2;\;-2)$.
Since $f'_y\neq 0$ it is possible to define an implicit function $x\to y\,(x)$.
Then we get, at point $(1,1)$: $\dfrac{\mathrm{d}y}{\mathrm{d}x}(1)=-\dfrac{f'_x(1,1)}{f'_y(1,1)}=-\dfrac{-2}{-2}=-1\,.$

II M 2) Solve the problem $\begin{cases}\text{Max/min } f(x,y)=x\,y\\ \text{u.c. : }\begin{cases}x^2-2x-y\leq 0\\ y-x\leq 0\end{cases}\end{cases}$.



The objective function of the problem is a continuous function, the constraint defines a feasible region which is a closed and bounded (compact) set, and so we can apply Weierstrass Theorem. Surely the function admits maximum value and minimum value.
To solve the problem we use the Kuhn-Tucker conditions.
We form the Lagrangian function:
$\Lambda(x,y,\lambda) = x\,y - \lambda_1\left(x^2 - 2x - y\right) - \lambda_2(y - x)\,.$
By applying the first order conditions we have:

1) case $\lambda_1 = 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = y = 0 \\ \Lambda'_y = x = 0 \\ x^2 - 2x - y \le 0 \\ y - x \le 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 - 0 - 0 \le 0 \\ 0 - 0 \le 0 \end{cases}.$$

Since $\mathbb{H}(x,y) = \mathbb{H}(0,0) = \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\|$ from $|\mathbb{H}_2| < 0$ it follows that $(0,0)$ is a saddle point.

2) case $\lambda_1 \ne 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = y - 2\lambda_1 x + 2\lambda_1 = 0 \\ \Lambda'_y = x + \lambda_1 = 0 \\ y = x^2 - 2x \\ y - x \le 0 \end{cases} \Rightarrow \begin{cases} x = -\lambda_1 \\ y + 2\lambda_1^2 + 2\lambda_1 = 0 \\ y = x^2 - 2x \\ y - x \le 0 \end{cases} \Rightarrow \begin{cases} x = -\lambda_1 \\ y = -2\lambda_1^2 - 2\lambda_1 \\ 2\lambda_1^2 + 2\lambda_1 + \lambda_1^2 + 2\lambda_1 = 0 \\ y - x \le 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = -\lambda_1 \\ y = -2\lambda_1^2 - 2\lambda_1 \\ 3\lambda_1^2 + 4\lambda_1 = 0 \\ y - x \le 0 \end{cases} \Rightarrow \begin{cases} x = -\lambda_1 \\ y = -2\lambda_1^2 - 2\lambda_1 \\ \lambda_1(3\lambda_1 + 4) = 0 \\ y - x \le 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ \lambda_1 = 0 \\ 0 \le 0 \end{cases} \cup \begin{cases} x = \frac{4}{3} \\ y = -\frac{8}{9} \\ \lambda_1 = -\frac{4}{3} \\ -\frac{8}{9} - \frac{4}{3} \le 0 \end{cases}.$$

We already know the nature of the point $(0,0)$ while $\left(\frac{4}{3}, -\frac{8}{9}\right)$, since $\lambda_1 = -\frac{4}{3} < 0$ could be a minimum point.

3) case $\lambda_1 = 0, \lambda_2 \ne 0$:

$$\begin{cases} \Lambda'_x = y + \lambda_2 = 0 \\ \Lambda'_y = x - \lambda_2 = 0 \\ y = x \\ x^2 - 2x - y \le 0 \end{cases} \Rightarrow \begin{cases} x = \lambda_2 \\ y = -\lambda_2 \\ \lambda_2 = -\lambda_2 \\ x^2 - 2x - y \le 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ \lambda_2 = 0 \\ 0 \le 0 \end{cases}.$$

We already know the nature of the point $(0,0)$.

4) case $\lambda_1 \neq 0, \lambda_2 \neq 0$:

$$\begin{cases} y = x \\ y = x^2 - 2x \end{cases} \Rightarrow \begin{cases} y = x \\ x^2 - 3x = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \cup \begin{cases} x = 3 \\ y = 3 \end{cases}.$$

$$\begin{cases} \Lambda'_x = y - 2\lambda_1 x + 2\lambda_1 + \lambda_2 = 0 \\ \Lambda'_y = x + \lambda_1 - \lambda_2 = 0 \\ x = 0 \\ y = 0 \end{cases} \Rightarrow \begin{cases} 2\lambda_1 + \lambda_2 = 0 \\ \lambda_1 - \lambda_2 = 0 \\ x = 0 \\ y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}$$

$$\begin{cases} \Lambda'_x = y - 2\lambda_1 x + 2\lambda_1 + \lambda_2 = 0 \\ \Lambda'_y = x + \lambda_1 - \lambda_2 = 0 \\ x = 3 \\ y = 3 \end{cases} \Rightarrow \begin{cases} 3 - 6\lambda_1 + 2\lambda_1 + \lambda_2 = 0 \\ 3 + \lambda_1 - \lambda_2 = 0 \\ x = 3 \\ y = 3 \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = 3 \\ \lambda_1 = 2 \\ \lambda_2 = 5 \end{cases}.$$

We already know the nature of the point $(0,0)$ while the point $(3,3)$, since $\lambda_1 > 0, \lambda_2 > 0$ could be a maximum point.

Having found only two solutions, one for the maximum and one for the minimum, these are the solutions of the problem, with $f(3,3) = 9$ and $f\left(\frac{4}{3}, -\frac{8}{9}\right) = -\frac{32}{27}$.

II M 3) Given the function $f(x,y) = x\,e^{y-x} - y\,e^{x-y}$, calculate $\mathcal{D}_v f(1,1)$, with v the unit vector of the vector from the point $(1,1)$ to the point $(2,2)$.

The function $f(x,y) = x\,e^{y-x} - y\,e^{x-y}$ is differentiable and so $\mathcal{D}_v f(P_0) = \nabla f(P_0) \cdot v$.
From $\nabla f(x,y) = (e^{y-x} - x\,e^{y-x} - y\,e^{x-y}; x\,e^{y-x} - e^{x-y} + y\,e^{x-y})$ we get :
$\nabla f(1,1) = (-1,1)$.
The vector that goes from the point $(1,1)$ to the point $(2,2)$ is $\mathbb{V} = (2,2) - (1,1) = (1,1)$ and so $v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\mathcal{D}_v f(1,1) = (-1,1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 0$.

II M 4) Given the function $f(x,y,z) = x^3\,e^{xy-z} + \log\left(2yz - x^2\right)$, calculate its gradient at the point $P_0 = (1,1,1)$.

Since:

$\dfrac{\partial f}{\partial x} = 3x^2\,e^{xy-z} + x^3 y\,e^{xy-z} + \dfrac{1}{2yz - x^2} \cdot (-2x)$;

$\dfrac{\partial f}{\partial y} = x^4\,e^{xy-z} + \dfrac{1}{2yz - x^2} \cdot (2z)$;

$\dfrac{\partial f}{\partial z} = x^3\,e^{xy-z}(-1) + \dfrac{1}{2yz - x^2} \cdot (2y)$;

it is $\nabla f(1,1,1) = (3 + 1 - 2; 1 + 2; -1 + 2) = (2,3,1)$.