Compito di Analisi Matematica del 11/11/2017 [CAM1]

IM1) $\frac{9}{i-1}+\frac{9}{1+i}=\frac{9}{1+i}-\frac{9}{1-i}=\frac{9-9i-9-9i}{(1+i)(1-i)}=\frac{-18i}{1+1}=-9i=+9\left(\cos\frac{3}{2}\pi+i\,\text{sen}\,\frac{3}{2}\pi\right)$.

$\sqrt[3]{z}=\sqrt[3]{9}\cdot\left(\cos\left(\frac{1}{2}\pi+K\cdot\frac{2\pi}{3}\right)+i\,\text{sen}\left(\frac{\pi}{2}+K\cdot\frac{2\pi}{3}\right)\right)$; $0\leq K\leq 2$. Se $k=0$: $\sqrt[3]{9}\cdot\left(\cos\frac{\pi}{2}+i\,\text{sen}\frac{\pi}{2}\right)=\sqrt[3]{9}\cdot i$;

Se $\underline{k=1}$: $\sqrt[3]{9}\cdot\left(\cos\frac{7}{6}\pi+i\,\text{sen}\,\frac{7}{6}\pi\right)=\sqrt[3]{9}\cdot\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$; Se $\underline{k=2}$: $\sqrt[3]{9}\cdot\left(\cos\frac{11}{6}\pi+i\,\text{sen}\,\frac{11}{6}\pi\right)=\sqrt[3]{9}\cdot\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$.

IM2) $f(x;y)=\begin{cases}\frac{x^6-y^6}{(x^2+y^2)^2} & : (x;y)\neq(0;0)\\ 0 & : (x;y)=(0;0)\end{cases}$. Per la continuità: $\lim\limits_{(x;y)\to(0;0)} f(x;y) \Rightarrow$

$\Rightarrow \lim\limits_{\rho\to 0}\frac{\rho^{\cancel{6}\,2}(\cos^6\vartheta-\text{sen}^6\vartheta)}{\cancel{\rho^4}}=0$. Convergenza Uniforme in quanto:

$|\rho^2(\cos^6\vartheta-\text{sen}^6\vartheta)|<2\rho^2<\varepsilon$ per $0<\rho<\frac{\sqrt{2\varepsilon}}{2}$.

$\frac{\partial f}{\partial x}(0;0)=\lim\limits_{h\to 0}\frac{1}{h}\left(\frac{h^6}{h^4}-0\right)=\lim\limits_{h\to 0}\frac{h^6}{h^5}=0$; $\frac{\partial f}{\partial y}(0;0)=\lim\limits_{h\to 0}\frac{1}{h}\left(\frac{-h^6}{h^4}-0\right)=\lim\limits_{h\to 0}-\frac{h^6}{h^5}=0$.

Differenziabilità: $\lim\limits_{(x;y)\to(0;0)}\frac{f(x;y)-f(0;0)-\nabla f(0;0)\cdot(x-0;y-0)}{\sqrt{x^2+y^2}}\overset{?}{=}0 \Rightarrow$

$\Rightarrow \lim\limits_{\rho\to 0}\frac{1}{\rho}\cdot\left(\frac{\rho^6(\cos^6\vartheta-\text{sen}^6\vartheta)}{\rho^4}-0-(0;0)(\rho\cos\vartheta;\rho\,\text{sen}\,\vartheta)\right)=\lim\limits_{\rho\to 0}\rho\cdot(\cos^6\vartheta-\text{sen}^6\vartheta)=0$

Dato che la convergenza è uniforme, la funzione è differenziabile.

IM3) $f(x;y)=xy-e^{y-x}$; $f(1;1)=1-1=0$. $\nabla f(x;y)=(y+e^{y-x};x-e^{y-x})$,

$\nabla f(1;1)=(2;0)$. Dato che $f'_x(1;1)=2\neq 0$ si può definire $x=x(y)$ come f. implicita.

$H(x;y)=\begin{Vmatrix}-e^{y-x} & 1+e^{y-x}\\ 1+e^{y-x} & -e^{y-x}\end{Vmatrix}$; $H(1;1)=\begin{Vmatrix}-1 & 2\\ 2 & -1\end{Vmatrix}$. $x''(1)=-\frac{f''_{xx}\cdot(x')^2+2f''_{xy}\cdot x'+f''_{yy}}{f'_x}=$

$=x''(1)=-\frac{(-1)\cdot 0+2\cdot 2\cdot 0+(-1)}{2}=\frac{1}{2}>0$. Dato che $x'(1)=0$ e $x''(1)=\frac{1}{2}>0 \Rightarrow y=1$ Punto di MIN.

IM4) Essendo $f(x;y)$ differenziabile: $D_v(f(x_0))=\nabla f\cdot v \Rightarrow$

$D_v f(x_0)=(f'_x;f'_y)\cdot\left(\frac{1}{\sqrt{2}};\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}(f'_x+f'_y)=2 \Rightarrow f'_x+f'_y=2\sqrt{2}$;

$D_w f(x_0)=(f'_x;f'_y)\cdot\left(\frac{1}{\sqrt{2}};-\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}(f'_x-f'_y)=-3 \Rightarrow f'_x-f'_y=-3\sqrt{2}$.

$$\begin{cases} f'_x + f'_y = 2\sqrt{2} \\ f'_x - f'_y = -3\sqrt{2} \end{cases} \Rightarrow \begin{cases} f'_y = 2\sqrt{2} - f'_x \\ f'_x + f'_x - 2\sqrt{2} = -3\sqrt{2} \end{cases} \Rightarrow \begin{cases} f'_x = -\frac{\sqrt{2}}{2} \\ f'_y = 2\sqrt{2} + \frac{\sqrt{2}}{2} = \frac{5}{2}\sqrt{2} \end{cases}.$$

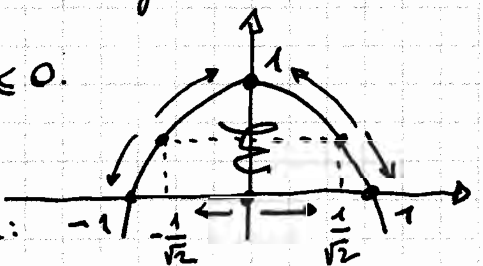
II M 1) $f(x;y) = x^2+y^2-xy^2-x$. $\nabla f(x;y) = \underline{0} \Rightarrow \begin{cases} f'_x = 2x - y^2 - 1 = 0 \\ f'_y = 2y - 2xy = 2y(1-x) = 0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x = \frac{1}{2} \\ y = 0 \end{cases} \cup \begin{cases} y^2 = 1 \\ x = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases} \cup \begin{cases} x = 1 \\ y = -1 \end{cases}$. Tre punti stazionari: $\left(\frac{1}{2};0\right); (1;1); (1;-1)$.

$H(x;y) = \left\| \begin{matrix} 2 & -2y \\ -2y & 2-2x \end{matrix} \right\|$. $H\left(\frac{1}{2};0\right) = \left\| \begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix} \right\| : \begin{cases} |H_1| > 0 \\ |H_2| = 2 > 0 \end{cases}$: Punto di minimo;

$H(1;1) = \left\| \begin{matrix} 2 & -2 \\ -2 & 0 \end{matrix} \right\| : |H_2| = -4 < 0$: P. di Sella; $H(1;-1) = \left\| \begin{matrix} 2 & 2 \\ 2 & 0 \end{matrix} \right\| : |H_2| = -4 < 0$: P. di Sella.

II M2) $\begin{cases} \text{Max/min } f(x;y) = x^2+y^2 \\ \text{s.v. } 0 \le y \le 1-x^2 \end{cases} \Rightarrow \begin{cases} \text{Max/min } f(x;y) = x^2+y^2 \\ \text{s.v. } \begin{cases} -y \le 0 \\ y+x^2-1 \le 0. \end{cases} \end{cases}$



La funzione è continua, Σ è limitato e chiuso, i vincoli sono qualificati. Usiamo la Lagrangiana:

$$\Lambda(x;y;\lambda_1;\lambda_2) = x^2+y^2 - \lambda_1(-y) - \lambda_2(y+x^2-1)$$

<u>Caso $\lambda_1 = \lambda_2 = 0$</u>

$$\begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = 2y = 0 \\ y \geq 0 \\ y \leq 1-x^2 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 \geq 0 \\ 0 \leq 1 \end{cases} \Rightarrow H(x;y) = H(0;0) = \left\| \begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix} \right\| \Rightarrow \begin{cases} |H_1| > 0 \\ |H_2| > 0 \end{cases} : \text{P. di Minimo.}$$

<u>Caso $\lambda_1 \neq 0; \lambda_2 = 0$</u>

$$\begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = 2y + \lambda_1 = 0 \\ y = 0 \\ y \leq 1-x^2 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ \lambda_1 = 0 \\ 0 \leq 1 \end{cases} : (0;0) \text{ già visto;}$$

Caso $\lambda_1 = 0;\ \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = 2x - 2\lambda_2 x = 2x(1-\lambda_2) = 0 \\ \Lambda'_y = 2y - \lambda_2 = 0 \\ y \geq 0 \\ y = 1 - x^2 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_2 = 2 > 0 \\ 1 \geq 0 \end{cases} \cup \begin{cases} \lambda_2 = 1 > 0 \\ y = \frac{1}{2} \\ x^2 = \frac{1}{2} \\ y \geq 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\sqrt{2}} \\ y = \frac{1}{2} \\ \lambda_2 = 1 > 0 \\ \frac{1}{2} \geq 0 \end{cases} \cup \begin{cases} x = -\frac{1}{\sqrt{2}} \\ y = \frac{1}{2} \\ \lambda_2 = 1 > 0 \\ \frac{1}{2} \geq 0 \end{cases}$$

Max ? Max ? Max ?

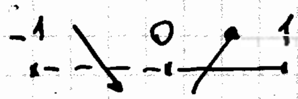
Caso $\lambda_1 \neq 0;\ \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = 2x - 2\lambda_2 x = 0 \\ \Lambda'_y = 2y + \lambda_1 - \lambda_2 = 0 \\ y = 0 \\ y = 1 - x^2 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \end{cases} \cup \begin{cases} x = -1 \\ y = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ \lambda_2 = 1 \\ \lambda_1 = 1 \end{cases} \cup \begin{cases} x = -1 \\ y = 0 \\ \lambda_2 = 1 \\ \lambda_1 = 1 \end{cases}.$$

Max ? Max ?

Analisi sulla frontiera di E.

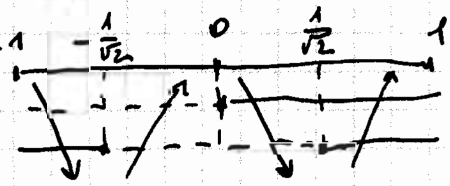
Se $y = 0 \Rightarrow f(x;0) = x^2 \Rightarrow f'(x) = 2x \geq 0$ per $x \geq 0$:



$(0;0)$ si conferma punto di minimo;

Se $y = 1 - x^2 \Rightarrow f(x;1-x^2) = x^2 + 1 + x^4 - 2x^2 = x^4 - x^2 + 1 \Rightarrow$

$\Rightarrow f'(x) = 4x^3 - 2x = 2x(2x^2 - 1) \geq 0$



$$\begin{cases} x \geq 0 \\ x^2 \geq \frac{1}{2} \end{cases} \Rightarrow x \leq -\frac{1}{\sqrt{2}} \cup x \geq \frac{1}{\sqrt{2}}$$

I punti $\left(-\frac{1}{\sqrt{2}};\frac{1}{2}\right)$ e $\left(\frac{1}{\sqrt{2}};\frac{1}{2}\right)$ risultano, sulla frontiera, punti di minimo mentre invece erano segnalati come possibili punti di Massimo. Quindi non sono né di massimo né di minimo.

In $(1;0)$ e $(-1;0)$ ci sono punti di massimo, con $f(1;1) = f(1;-1) = 2$.

Nel punto $(0;0)$, con $f(0;0) = 0$, c'è il punto di minimo.

II M3) $y''-y'=0 \Rightarrow t^2-t=t(t-1)=0 \Rightarrow t=0$ e $t=1$. Soluzione Omogenea: $y=c_1+c_2e^x$.

$y_0=a\,\mathrm{sen}\,x+b\cos x$; $y_0'=a\cdot\cos x-b\,\mathrm{sen}\,x$; $y_0''=-a\,\mathrm{sen}\,x-b\cos x \Rightarrow$

$\Rightarrow y''-y'=-a\,\mathrm{sen}\,x-b\cos x-a\cdot\cos x+b\,\mathrm{sen}\,x=\mathrm{sen}\,x \Rightarrow \begin{cases} b-a=1\\ -a-b=0\end{cases} \Rightarrow \begin{cases} a=-\frac{1}{2}\\ b=\frac{1}{2}\end{cases}.$

Soluzione Generale: $y=c_1+c_2e^x-\frac{1}{2}\mathrm{sen}\,x+\frac{1}{2}\cos x$. $\begin{cases} y(0)=c_1+c_2+\frac{1}{2}=1\\ y'(0)=0+c_2-\frac{1}{2}=1\end{cases} = \begin{cases} c_1=\frac{1}{2}-c_2=-1.\\ c_2=\frac{3}{2}.\end{cases}$

II M4) $\int_0^4\int_{(x-1)^2}^{2x+1}(x^2+y)\,dy\,dx=\int_0^4\left(x^2y+\frac{y^2}{2}\right|_{(x-1)^2}^{2x+1}dx=\int_0^4 4x^3-x^4+2x^2+2x+\frac{1}{2}-\frac{1}{2}(x-1)^4\,dx=$

$=\left(x^4-\frac{1}{5}x^5+\frac{2}{3}x^3+x^2+\frac{1}{2}x-\frac{1}{10}(x-1)^5\right|_0^4=\frac{263}{3}.$