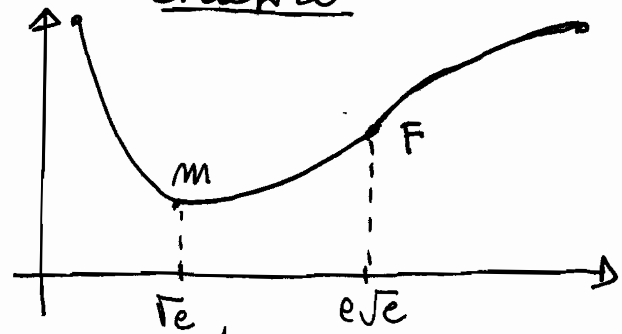
Compito di Matematica Generale del 11/1/2022 $\boxed{\text{CMG 1}}$

1) $f(x)=\log^2 x-\log x+1$. C.E.: $x>0$. $\lim_{x\to0^+} f(x)=+\infty$; $\lim_{x\to+\infty} f(x)=+\infty$.

$\log^2 x-\log x+1\geq 0$: $\log x=\frac{1\pm\sqrt{1-4}}{2}\Rightarrow f(x)>0\ \forall x>0$. $f(\sqrt e)=\frac34$.

Grafico



$f'(x)=2\cdot\log x\cdot\frac1x-\frac1x=\frac1x(2\log x-1)\geq 0$

per $\log x\geq\frac12\Rightarrow x\geq e^{\frac12}=\sqrt e$

$f''(x)=-\frac1{x^2}\cdot(2\log x-1)+\frac2{x^2}=\frac1{x^2}\cdot(3-2\log x)\geq0$

per $\log x\leq\frac32\Rightarrow x\leq e^{\frac32}=e\sqrt e$

2) $\lim_{x\to0}\frac{3^{\operatorname{sen}x}-2^x}{\log(1+x)}=\lim_{x\to0}\left(\frac{3^{\operatorname{sen}x}-1}{\operatorname{sen}x}\cdot\underbrace{\frac{\operatorname{sen}x}{x}}_{\to1}-\frac{2^x-1}{x}\right)\cdot\underbrace{\frac{x}{\log(1+x)}}_{\to1}=\log3-\log2=\log\frac32$.

$\lim_{x\to+\infty}\left(\frac{4-3x+x^2}{5+3x+2x^2}\right)^{1-\sqrt x}=\left(\to\frac12\right)^{(\to-\infty)}=+\infty$.

3) $f(x)=\begin{cases}x^2-x : x<0\\ e^{kx}-1 : x\geq0\end{cases}$. $f(0)=1-1=0$; $\lim_{x\to0^-}f(x)=0=\lim_{x\to0^+}f(x)=f(0)\ \forall k$.

$f'(x)=\begin{cases}2x-1 : x<0\\ ke^{kx} : x>0\end{cases}$. $\lim_{x\to0^-}f'(x)=-1$; $\lim_{x\to0^+}f'(x)=k$. Quindi $f(x)$ è derivabile anche in $x=0$ se $k=-1$.

4) $f(x)=\log_2\left(\frac{e^x-1}{e^x+1}\right)$. C.E.: $e^x-1>0\Rightarrow x>0$. $\lim_{x\to0^+}f(x)=-\infty$; $\lim_{x\to+\infty}f(x)=\log_2 1=0$.

$f'(x)=\frac{e^x+1}{e^x-1}\cdot\log_2 e\cdot\frac{e^x(e^x+1)-e^x(e^x-1)}{(e^x+1)^2}\Rightarrow\frac{2e^x}{e^{2x}-1}>0\ \forall x>0$. Invertibile in tutto il C.E. in quanto strettamente crescente. $f:\,]0;+\infty[\,\to\,]-\infty;0[$.

$f^{-1}:\,]-\infty;0[\,\to\,]0;+\infty[$. $y=\log_2\left(\frac{e^x-1}{e^x+1}\right)\Rightarrow\frac{e^x-1}{e^x+1}=2^y\Rightarrow e^x-1=2^y\cdot e^x+2^y\Rightarrow$

$\Rightarrow e^x(1-2^y)=1+2^y\Rightarrow x=\log\left(\frac{1+2^y}{1-2^y}\right)$. Inversa: $f^{-1}(x)$: $y=\log\left(\frac{1+2^x}{1-2^x}\right)$.

5) $\int_0^1 x\,e^{2x}-e^{-x}\,dx\Rightarrow\int xe^{2x}-e^{-x}dx=\frac12xe^{2x}-\int\frac12e^{2x}dx+e^{-x}=\frac12xe^{2x}-\frac14e^{2x}+e^{-x}\Rightarrow$

CMG 2

$$\Rightarrow \int_0^1 x e^{2x} - e^{-x}\, dx = \left(\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + e^{-x}\right|_0^1 = \left(\frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{e}\right) - \left(0 - \frac{1}{4} + 1\right) = \frac{1}{4}e^2 + \frac{1}{e} - \frac{3}{4}.$$

6) $df(x_0) = f'(x_0)\cdot dx \Rightarrow (6x_0 - 2)\cdot \frac{1}{2} = 4 \Rightarrow 3x_0 - 1 = 4 \Rightarrow 3x_0 = 5 \Rightarrow x_0 = \frac{5}{3}.$

7) $f(x;y) = x^3 - x^2 + y e^y$. $\nabla f(x;y) = (0;0) \Rightarrow$

$$\Rightarrow \begin{cases} f'_x = 3x^2 - 2x = x(3x-2) = 0 \\ f'_y = e^y + y e^y = e^y(1+y) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -1 \end{cases} \cup \begin{cases} x = \frac{2}{3} \\ y = -1 \end{cases}. \quad H(x;y) = \begin{Vmatrix} 6x-2 & 0 \\ 0 & e^y(y+2) \end{Vmatrix}$$

$$H(0;-1) = \begin{Vmatrix} -2 & 0 \\ 0 & \frac{1}{e} \end{Vmatrix} : |H_2| = -\frac{2}{e} < 0 : \text{P. Sella};\ H\left(\frac{2}{3};-1\right) = \begin{Vmatrix} 2 & 0 \\ 0 & \frac{1}{e} \end{Vmatrix} : \begin{cases} |H_1| = 2 > 0;\ \frac{1}{e} > 0 \\ |H_2| = \frac{2}{e} > 0 \end{cases} : \text{P. MINIMO}.$$

8) $$A\cdot X = \begin{Vmatrix} 1 & -1 & K \\ 0 & K & 1 \end{Vmatrix} \cdot \begin{Vmatrix} K \\ 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} K - 1 + K \\ 0 + K + 1 \end{Vmatrix} = \begin{Vmatrix} 2K-1 \\ K+1 \end{Vmatrix}.$$

$$\|2K-1; K+1\| = \sqrt{2} \Rightarrow \sqrt{(2K-1)^2 + (K+1)^2} = \sqrt{2} \Rightarrow 5K^2 - 2K + 2 = 2 \Rightarrow K(5K-2) = 0 < \begin{matrix} K = 0 \\ K = \frac{2}{5} \end{matrix};$$

$$(2K-1; K+1) \,//\, (2;3) \Rightarrow \frac{2K-1}{2} = \frac{K+1}{3} \Rightarrow 6K - 3 = 2K + 2 \Rightarrow 4K = 5 \Rightarrow K = \frac{5}{4};$$

$$(2K-1; K+1) \perp (2;3) \Rightarrow (2K-1; K+1)\cdot(2;3) = 0 \Rightarrow 4K - 2 + 3K + 3 = 7K + 1 = 0 \Rightarrow K = -\frac{1}{7}.$$

9)

A B C	(A e B)	non C	[(A e B) ⇔ non C]	(C ⇒ B)	[(A e B) ⇔ non C] o (C ⇒ B)	(A ⇒ C)	
1 1 1	1	0	0	1	1	1	
1 1 0	1	1	1	1	1	0	*
1 0 1	0	0	1	0	1	1	
1 0 0	0	1	0	1	1	0	*
0 1 1	0	0	1	1	1	1	
0 1 0	0	1	0	1	1	1	
0 0 1	0	0	1	0	1	1	
0 0 0	0	1	0	1	1	1	

10) $$F(x) = e^{f(x)} \Rightarrow F'(x) = f'(x)\cdot e^{f(x)} \Rightarrow F''(x) = f''(x)\cdot e^{f(x)} + [f'(x)]^2 \cdot e^{f(x)}$$

$$\Rightarrow (+)\cdot(+) + (+)\cdot(+) \Rightarrow F''(x) \geq 0\ \forall x$$

dato che $f''(x) \geq 0$ e $[f'(x)]^2 \geq 0$ per le ipotesi date.