Part A: Questions with extended solutions about expected utility, risk and insurance

1.

Consumer A and B initial wealth is zero. Their preferences for non-contingent consumption are represented by the utility functions: $u_A(c_A) = 2c_A$ $\quad u_B(c_B) = c_B^{1/4}$

Contingent consumption is produced by the lottery L, with prizes L1 ≠ L2, conditional on state 1 and 2, occuring with probability π and (1 - π), respectively. Preferences for contingent consumption satisfy the expected utility property.
Which of the following statements is wrong?

a. Risk aversion is lower for A than for B
b. $| MRS^A | < | MRS^B |$ if $c_1^B < c_2^B$
c. $| MRS^A | > | MRS^B |$ if $c_1^B > c_2^B$
d. A's preferences for contingent consumption are weakly convex, B's preferences are strictly convex
e. The certainty equivalent of lottery L is smaller for A than for B
f. None

2.

Consumer A initial wealth is W = 8. His preferences for non-contingent consumption are represented by the utility function: $u_A(c_A) = \log(c_A)$.
A can get a fraction α of a lottery L, where $1 \geq \alpha \geq 0$ and L offers prizes $L_1 = -4$ with probability ½ and $L2 = 8$ with probability ½.
Determine the optimal stake α in the lottery.

The contingent consumption corresponding to a stake α of the lottery is
$c_1(\alpha) = W_1 - 4\alpha$
$c_2(\alpha) = W_2 + 8\alpha$
The budget constraint expressing the possibility of consumption transfer between states 1 and 2, through variation of the stake α, is expressed by the relation:

$$\frac{[c_2(\alpha) - w_2]}{[c_1(\alpha) - w_1]} = \frac{8\alpha}{-4\alpha} = -2$$

The slope of the budget line is therefore - 2.
The expected utility of A is $EU_A = ½\, u_A(c_1(\alpha)) + ½\, u_A(c_2(\alpha))$. Recalling that u (c) = log (c) and that the derivative of the log (c) is 1 / c, we can write:

$$| MRS_A | = [½\, u'_A(c_1(\alpha))] / [½\, u'_A(c_2(\alpha))] = (c_2(\alpha)) / (c_1(\alpha)) = \frac{8+8\alpha}{8-4\alpha}$$

Because preferences are strictly convex, the necessary and sufficient condition for a non-corner solution to consumer's expected-utility maximization problem is:

$$| MRS | = | \text{Slope of the budget line} | \text{ that is } \frac{8+8\alpha}{8-4\alpha} = 2$$

$2(1 + \alpha) = 2\,(2 - \alpha)$
$(1 + \alpha) = (2 - \alpha)$

$2\alpha = 1 \qquad \alpha = ½$

Exercise 3.

The initial wealth of consumers A and B is zero. Their preferences for non contingent consumption are represented by the utility functions: $u_A(c_A) = 2c_A$ $u_B(c_B) = c_B^{1/2}$. A and B divide the prize of a lottery L, offering prize $L_1 = 4$ with probability ½, and prize $L_2 = 16$ with probability ½. Which of the following prize shares to B (and the remainder to A) corresponds to a Pareto efficient allocation of risk between A and B?

a. $L1_B = 1$ $L2_B = 13$
b. $L1_B = 2$ $L2_B = 8$
c. $L1_B = 3$ $L2_B = 3$
d. None

A is risk neutral, B is risk averse. If the division of risk is Pareto efficient, all risk must fall upon A. Therefore the correct answer is c.

Exercise 4

Consumer A has initial wealth W = 1200 Euro, his preferences for contingent consumption meet the expected utility form, and his utility function for non contingent consumption is $u(c) = c^{½}$. A has the opportunity to choose between a sure money prize C, or an investment L of 300 Euro, yielding Euro 700, if state G obtains, and zero, if state B obtains. The probability of states G and B is ½, ½. Which of the following conditions will necessarily induce A to accept investment L?

a. C <25
b. C <35
c. C> 25
d. C <40
e. None

The expected utility of investment L is:

$EU(W+L) = ½[c_G]^{1/2} + ½[c_B]^{1/2} =$

$½(1200 - 300 + 700)^{1/2} + ½(1200 - 300)^{1/2} = ½\, 40 + ½\, 30 = 35$

The certainty equivalent (CE) of investment L is the sum x such that

$u(W + x) = (W + x)^{1/2} = EU(W + L) = 35$

$W + x = 35^2 = 1225$

$CE(L) = x = 1225 - W = 25$

In case c, A prefers the sure prize C to L. This may also happen in cases b and d.

The correct answer is a.

Exercise 5

Consumer A has initial wealth W = 121 euro, her preferences for contingent consumption meet the expected utility form, and her utility function for non contingent consumption is $u(c) = 2c^{½}$.

She has the opportunity to participate in a lottery L, offering the prize 23 euro with probability ½, and the negative prize − 21 euro with probability ½.

Defining αL the lottery that offers 23α with probability ½ and (-21α) with probability ½, indicate which of the following shares of L consumer A will decide to accept and why.

a. α = 1
b. α = 0
c. α = 2
d. 0 <α <1
e. none of the other answers

Notice that the share α = 1 gives rise to the contingent consumption (W + L), the share α = 0 produces the contingent consumption W (such that W1 = W2).
The expected utility of accepting the share α of the lottery is:
$EU(W + \alpha L) = ½\ 2[121 + \alpha 23]^{1/2} + ½\ 2[121 - \alpha 21]^{1/2} = [121 + \alpha 23]^{1/2} + [121 - \alpha 21]^{1/2}$

Simple calculations show that EU (W + L) = U (W); the consumer is indifferent between the share α = 1 and the share α = 0. That is, the certainty equivalent of W + L is CE = W.
Values of α such that 0 <α <1 produce bundles of contingent consumption that are intermediate between baskets W and W + L. Because the consumer is risk averse, she has strictly convex preferences for contingent consumption. The consumer is certainly willing to accept shares of L in the range 0 <α <1, because they correspond to the baskets of contingent consumption strictly preferred both to W and W + L.
The consumer is definitely not ready to accept shares of L such that α > 1, because these shares carry a higher risk than is implied by L, that is, they represent 'more extreme' baskets. Therefore the correct answer is d.

Exercise 6

Anna's wealth is $W_1 = 200$ if state 1 obtains, $W_2 = 400$ if state 2 obtains. The two states occur with probability ¼, ¾. Anna's utility function for contingent consumption is $U(c) = 3c^{1/2}$ and the money price of good c is identical (p = 1) in the two states. Paying an insurance premium γ = ¼ for each unit of insured wealth, Anna is able to transfer wealth from the state 2 to the state 1 at the rate: $|dW2 / dW1| = \gamma / (1 - \gamma) = 1/3$. If Anna buys the optimal amount of insurance, how large is her consumption in state 1?

a. 175
b. 225
c. 375
d. 360
e. none of the other answers

solution: the probability of the state in which the damage occurs is ¼, that turns out to be equal to the premium γ = ¼ paid for each unit of insured wealth. Therefore the premium is fair and, because Anna is risk averse, her optimal choice is to buy insurance K = D (the value of damage). That is:
K = D = W2 – W1 = 200. Hence, C1 = W1 + K – γK = 350 =C2 = W2 – γK. The answer is e.

Exercise 7

A and B have equal wealth 400 Euro. Their utility functions for non contingent consumption are $U_A = c^{1/2}$, and $U_B = c$, respectively. Their preferences for contingent consumption meet the expected utility form. A has the opportunity to participate in a lottery L, producing the loss € − 300 , if state 1 occurs, and the prize € 9600, if state 2 occurs. States 1 and 2 occur with probability 9/10, and 1/10, respectively. A rationally decides she doesn't want to bear the full risk of L, but gives L to B. In return, B will give A €600, if state 2 obtains, and €x, if state 1 obtains. How large must be the payment x, in order that the risk allocation between A and B is Pareto efficient?

a) 0

b) 600

c) 400

d) 250

e) none of the other answers

Solution: A is risk averse, B is risk-neutral. Since a Pareto efficient allocation of risk involves $MRS_A = MRS_B$, it follows that all risk is borne by B, while A does not bear any; that is $C_1^A = C_2^A$: A's consumption in state 1 must be equal to A's consumption in the state 2.

Since $C_2^A = 400 + 600$; $C_1^A = 400 + x$; *therefore, $x = 600$. Actually, from $MRS_A = MRS_B$ we get:*

$$MRS_A = \frac{\frac{9}{10}\cdot 1/2\cdot\left(C_1^A\right)^{-1/2}}{\frac{1}{10}\cdot 1/2\cdot\left(C_2^A\right)^{-1/2}} = MRS_B = \frac{\frac{9}{10}}{\frac{1}{10}}$$

which implies $C_1^A \,/\, C_2^A = 1$.

Exercise 8

Anna's wealth is $W_b = 0$, if state b (bad) occurs, and $W_g = 1000$, if state g (good) occurs. The states b and g occur with probability 1/3, 2/3, respectively. Anna's utility function for sure wealth W is $u(W) = W^{1/2}$. Having the opportunity to insure a variable amount of wealth K against the risk of damage $D = W_g - W_b = 1000$, paying insurance premium γK, with $\gamma = ½$, Anna decides to ensure:

a. K = 1000
b. K = 800
c. K = 600
d. K = 400
e. None of the other answers is correct

As we learn from the treatment of insurance in Chapter 12 of Varian, Microeconomics:

- an agent facing the opportunity to insure a variable money sum K, paying a premium Kγ, has the opportunity to transfer wealth from the 'favorable' state to the 'unfavorable' state, along a straight budget line with slope: ***$-\gamma / (1 - \gamma)$***

Since $\gamma = ½ > = 1/3$ = probability of the adverse event, the risk averse agent is not willing to insure the full amount D.

Anna's contingent consumption is modified by K according to:

$C_b = 0 + K - \gamma K$; $\qquad C_g = 1000 - \gamma K$

Anna's marginal rate of substitution at (C_b, C_g) (slope of the indifference curve through (C_b, C_g)) is:

MRS = $-$ *UM* (C_{bad}) */ UM* (C_{good})

MRS is obtained as the ratio between the partial derivatives of *the expected utility* *with respect to* C_{bad} *and* C_{good}*, respectively.*

$$MRS = -\frac{(1/3)\cdot(1/2)(K-\gamma K)^{-1/2}}{(2/3)\cdot(1/2)(1000-\gamma K)^{-1/2}} = -\frac{(K-\gamma K)^{-1/2}}{2(1000-\gamma K)^{-1/2}}$$

Because Anna's preferences for contingent consumption are strictly convex (she is risk averse), the necessary and sufficient condition for an optimum choice of insurance is: MRS = - γ */ (1 -* γ*)*

$$MRS = -\frac{(1000-\gamma K)^{1/2}}{2(K-\gamma K)^{1/2}} = -\frac{\gamma}{(1-\gamma)} = -\frac{1/2}{1/2} = -1$$

$$(1000-\gamma K)^{1/2} = 2(K-\gamma K)^{1/2}$$

$$1000-\gamma K = 4(K-\gamma K) = 4K-2K = 2K$$

$$1000 = (5/2)K$$

$$K = 400$$

Exercise 9

There is a probability p = ¼ that A's wealth W = 12000, suffers a damage D = 11000.

A can ensure the wealth K $\leq$ W paying a premium γK, where the premium γ is fixed by the insurance company. Assume γ = ½. A's utility function for non-contingent consumption is $U^A = 4c^{1/2}$. A's preferences for contingent consumption satisfy the expected utility property. What is the amount K of insurance chosen by A?

If c_b *is consumption contingent on event b, where damage D occurs, and* c_g *is consumption contingent on event g (the damage-free event), the risky bundle* (W_b, W_g) *= (1000, 12000) corresponds to the decision K = 0. Ensuring K > 0, and paying a premium* γK*, the agent can transfer purchasing power from state g to state b.*

$c_g = W_g - \gamma K$

$c_b = W_b + K - \gamma K = W_b + (1-\gamma)K$

$c_g - W_g = -\gamma K$ (1)

$c_b - W_b = (1-\gamma)K$ (2)

dividing equations (1) and (2):

$(c_g - W_g) / (c_b - W_b) = -\gamma K / (1-\gamma)K = -\gamma / (1-\gamma) = -1$ ***slope of budget line***

Since γ *= ½ > probability of damage, the premium is not fair and the agent will ensure a value lower than D.*

The necessary and sufficient condition for an internal optimal choice of insurance is:

$MRS = -\gamma / (1-\gamma)$

$|MRS| = [¼ \bullet 2 \bullet c_b^{-1/2}] / [3/4 \bullet 2 \bullet c_g^{-1/2}] = 1$

$[c_b^{-1/2}] / [3 \bullet c_g^{-1/2}] = 1$ $c_g^{1/2} / 3 \bullet c_b^{1/2} = 1$ $c_g^{1/2} / c_b^{1/2} = 3$

$c_g / c_b = 9 \qquad\qquad c_g = 9\, c_b$

from the budget constraint $(c_g - W_g) / (c_b - W_b) = -1$
$(c_g - 12000) / (c_b - 1000) = -1$
$c_g - 12000 = 1000 - c_b$
$c_g + c_b = 13000 \qquad 9\, c_b + c_b = 10\, c_b = 13000 \qquad \boldsymbol{c_b = 1300}$

$c_b = 1300 = W_b + (1 - \gamma)K = 1000 \bullet ½\, K$
$½\, K = 1300 – 1000 = 300 \qquad\qquad K = 600$

Exercise 10.1
A and B have sure initial wealth $W_A = 100$ e $W_B = 0$, respectively. They agree to share a lottery L, paying prizes L1 = 400 with probability $\pi 1 = 1/3$ and L2 = 250 with probability $\pi 2 = 2/3$. As a result, A's wealth is $W_1^A = W_A + L_1^A$ if state 1 occurs, and $W_2^A = W_A + L_2^A$, if state 2 occurs; B's wealth is $W_1^B = L_1^B$, if state 1 occurs, and $W_2^B = L_2^B$, if state 2 occurs. The obvious constraints are $L_1^A + L_1^{AB} = 400$, $L_2^A + L_2^{AB} = 250$. Knowing that A is risk neutral, and B is risk averse, is a prize allocation such that $L_1^A = 150$ e $L_2^A = 0$ Pareto efficient?

Since A is risk neutral his preferences for non contingent consumption can be represented by a linear utility function of the type $u_A(c_A) = c_A$. It follows that his expected utility for contingent consumption has the form $U^A = \pi_1 c_1 + \pi_2 c_2$. $MRS_A = \pi_1 / \pi_2 = (1/3) / (2/3) = ½$.

The necessary condition for a Pareto efficient allocation of risk between A and B is:

$½ = | MRS_A | = | MRS_B |$. Because B is risk averse, the condition $| MRS_B | = ½$ obtains only if his contingent consumption bundles is such that $c_1^B = c_2^B$, that is, if he is not bearing any risk. All risk must be borne by A. If A's prizes are $L_1^A = 150$ e $L_2^A = 0$, B's prizes are $L_1^B = 250 = W_1^B$, and $L_2^B = 250 = W_2^B$. Thus he is not taking any risk and prize allocation is Pareto efficient.
A's contingent wealth is: $W_1^A = W_A + L_1^A = 250$ with probability 1/3, and $W_2^A = W_A + L_2^A = 100$ with probability 2/3.

Exercise 10.2
Assume that the lottery in the previous exercise is initially offered to B (risk averse), who decides to sell it to A (risk neutral). What is the maximum price P that A is willing to pay for L?

You must identify A's reservation price for the lottery, that is, the price P at which A is indifferent between buying or not-buying L. This price is defined by:

$EU_A = 1/3 \bullet u_A(W_A - P + L_1) + 2/3 \bullet u_A(W_A - P + L_2) = u_A(W_A)$

Recalling that: $W_A = 100 \quad L_1 = 400 \quad L_2 = 250 \quad u_A(c_A) = c_A$, *we obtain:*

$u_A(100) = 100 = 1/3 \bullet (100 - P + 400) + 2/3 \bullet (100 - P + 250)$

$P = [500/3 \quad + \quad 700/3] - 100 \quad = \quad 300$

If A buys L at the reservation price P, he gets $W_1^A = 200$, $W_2^A = 50$

B gets $W_1^B = 300$ $W_2^B = 300$. Because A is risk neutral, A's reservation price for L is $P = E(L) = (1/3)\,400 + (2/3)\,250 = 300$. Agent B gets sure wealth $E(L) = 300$, and being risk averse,
$E(L) > CE_B(L)$ = certainty equivalent of L for B.
Agent B gets a positive gain from the sale of L to agent A at price P, quantified by:
$E(L) - CE_B(L)$.

Exercise 11. The value of information on financial market (Varian Chapter 12)

Faced with the recent economic crisis, a family reconsiders its investment portfolio, and plans to invest the revenue from the sale of a real estate. The family considers two investment opportunities, a deposit C, yielding a sure monthly return of 400 euro, and a risky financial activity A, yielding a monthly return of €2000, if the market is 'bullish' (which happens with probability ½), and €100, if the market is 'bearish' (which happens with probability ½).

The family has a utility function for sure wealth $U(W) = 60 - 4000/W$. What is the choice suggested by maximization of expected utility, if the family doesn't have any other source of wealth?

The monthly utility u(C) is independent from any swing in financial markets $U(400) = 50$.

Monthly utility of A depends on market conditions. $EU(A) = ½\,U(2000) + ½\,U(100) = 39$.

(Notice that the expected value of A, is $E(A) = 1050$, but the certainty equivalent of A is the sum x such $U(x) = 39$, that is $CE(A) \approx 190.48 < 400$, because the family is strongly risk averse).

Still attracted by the possibility of a higher monthly return, the family considers the opportunity of gathering information on financial markets. It will then reconsider its investment on a monthly basis, investing either on C or on A, depending on the coming news about next monthly returns. The monthly newsletter N of a very reliable research center on financial market costs 10 euro, and will tell the family whether next month returns will be high or low.

Do you expect that the family is willing to purchase the newsletter?

The subscription may give two outcomes, both with probability ½. You may discover that the market is bearish, in which case you invest in C, getting $U(400-10) = 49.7$. Or you may find that the market is bullish, in which case you invest in A, getting $U(2000-10) = 57.9$. The expected utility of purchasing information is $EU(N) = ½\ 49{,}7 + ½\ 57.9 = 53.9$.

Comparing this value with the utility $U(400) = 50$ of the best choice, when the opportunity of information gathering is not available, the family decides to subscribe to the newsletter N.

Another method for checking whether the family will, or will not, purchase the newsletter is the comparison between the actual price $p = 10$ and the maximum price p_R the family is willing to pay.

To determine p_R , notice that it is convenient to buy information at price p, if the resulting expected utility is greater than or equal to the one you can get without buying information. So p_R is determined by the condition: ½ U(400 – p_R) + ½ U (2000 - p_R) = 50.

Finally, notice that a piece of information has no economic value for an agent, if its purchase cannot influence her decision. For example, if the returns form A, with high and low financial markets, were 2000 and 410, regardless of what monthly information the family can get by reading the newsletter, it will never decide to invest in C, and that information would be worthless.

Part B.- Examples of questions with solution

1. Agent A may participate in the lottery that assigns prize $c_1 = 64$ with probability 1/2 and prize c2 = 16 with probability 1/2. A's utility function for non-contingent consumption c is: $U(c) = 3c^{1/2}$. Indicate the expected value of the lottery and the expected utility of the lottery.

a) VA = 18, U (L) = 18
b) VA = 40, U (L) = 40
c) VA = 40, U (L) = 18
d) VA = 18, U (L) = 40
e) none of the other answers is correct

2. What is meant by *certainty equivalen*t of a lottery?
a) the expected value of the lottery
b) the expected utility of the lottery
c) the amount of money whose utility is equal to the lottery's expected utility
d) the utility of the expected value of the lottery
e) none of the other answers is correct

3. An agent with utility function $U(c) = c^{1/2}$ can choose whether receiving a sum c with certainty, or participate in the lottery that assigns prize $c_1 = 64$ with probability 1/2 and prize $c_2 = 16$ with probability 1/2. What is his decision?
a) he accepts the lottery if $c < 40$
b) he rejects the lottery, whatever *c* may be offered
c) he accepts the lottery anyway, because he is risk loving
d) he accepts the lottery if $c < 36$
e) none of the other answers is correct

4. What is the *risk premium* of a lottery?
a) the satisfaction an agent receives from participation in the lottery
b) the price to pay for participating in the lottery
c) the difference between the utility of the expected value of the lottery, and the expected utility of the lottery
d) the difference between the expected value of the lottery, and its certainty equivalent
e) none of the other answers is correct

5. A computer salesman is waiting for the arrival of a shipment of microchips from many days. He believes that there is now only a 25% chance that tomorrow the shipment will eventually arrive.
In case the shipment actually arrives tomorrow, he will carry out his business plans and earn w_1 = 160000. Otherwise he will fail to comply with the contract, and will earn $w_2 = 0$. The agent has a Von Neumann-Morgenstern expected utility function, and his utility for sure wealth w is $U(w) = w^{1/2}$. Today, a friend offers him to buy the rights to the shipment. What is the minimum price at which the agent will be willing to sell the rights?
a) 160000
b) 10000
c) 100
d) 400
e) none of the other answers is correct

6. The seller of the exercise 5 is very concerned about completing his business plans (remember that the probability that the shipment will not arrive is 75%). He then considers the possibility of buying an insurance against this event, in which case a sum K = 160000 would be covered by insurance, and he would pay a premium γK. What is the maximum insurance premium γK he would be willing to pay?

a) 160000
b) 150000
c) 10000
d) 400
e) none of the other answers is correct

SOLUTIONS

1. C
2. C
3. D
4. D
5. B
6. B

The initial wealth of A and B is zero.

Their utility functions for non-contingent consumption are $u_A = 2c^{1/2}$ *and* $u_B = 4c^{1/4}$.

A's and B's preference for contingent consumption meet the expected utility property.

A and B participate in the lottery L, splitting prizes L1 = 300, L2 = 1600, that are conditional on events 1 and 2, occurring with probability 1/3, and 2/3, respectively.

The prize shares of A and B satisfy: L1 = L1a + L1b; L2 = L2a + L2b.

The cost to participate in the lottery is 0, the sharing of risk is Pareto efficient, and L2a / L1a = 8.

Determine the prize shares L1a, L1b, L2a, L2b.

Pareto efficiency condition: $MRS^A = MRS^B$

$$MRS^A = \frac{1/3(L1A)^{-1/2}}{2/3(L2A)^{-1/2}} = (1/2)\frac{L2A^{1/2}}{L1A^{1/2}} =$$

$$MRS^B = \frac{1/3(L1B)^{-3/4}}{2/3(L2B)^{-3/4}} = (1/2)\frac{L2B^{3/4}}{L1B^{3/4}}$$

hence

$$\frac{L2B^{3/4}}{L1B^{3/4}} = \left(\frac{L2B}{L1A}\right)^{3/4} = \frac{L2A^{1/2}}{L1A^{1/2}} = 8^{1/2} = 2^{3/2}$$

because

$$L2A = 8L1A$$

that is

$$\left(\frac{L2B}{L1A}\right)^{3/4} = 2^{3/2}$$

the power 2/3 of the above equality is :

$$\left(\frac{L2B}{L1B}\right)^{1/2} = 2 \text{ cioè } \left(\frac{L2B}{L1B}\right) = 4$$

L1A + L1B = 300

L2A + L2B = 1600 = 8L1A + 4L1B

400 = 2L1A + L1B = 2L1A + 300 - L1A

L1A = 100 L2A = 800

L1B = 200 L2B = 800