Choice under risk: supplementary notes to H. Varian, Microeconomics, Chp. 12.

1. Risk aversion

Figure 1 shows the graph of a utility function for money income of a risk averse consumer. It is assumed, for simplicity, that income is spent on a consumption good c with price $p_c = 1$. The function *u (x)* is such that $u'(x) > 0$ (monotone preferences), and marginal utility is decreasing ($d^2u/d^2x<0$).

The basket (**x** - ε, **x** + ε) is a *contingent consumption bundle,* depending on the realization of a random event, with two possible mutually exclusive outcomes B and G, occuring with probability πB e πG, respectively, where πB + πG = 1. The situation described is equivalent to ownership of a lottery ticket yielding **x** + ε with probability π_G , and ***x*** − ε with probability π_B. The contingent bundle has average value x, given a chance event realization that, in figure, is assumed π = ½.

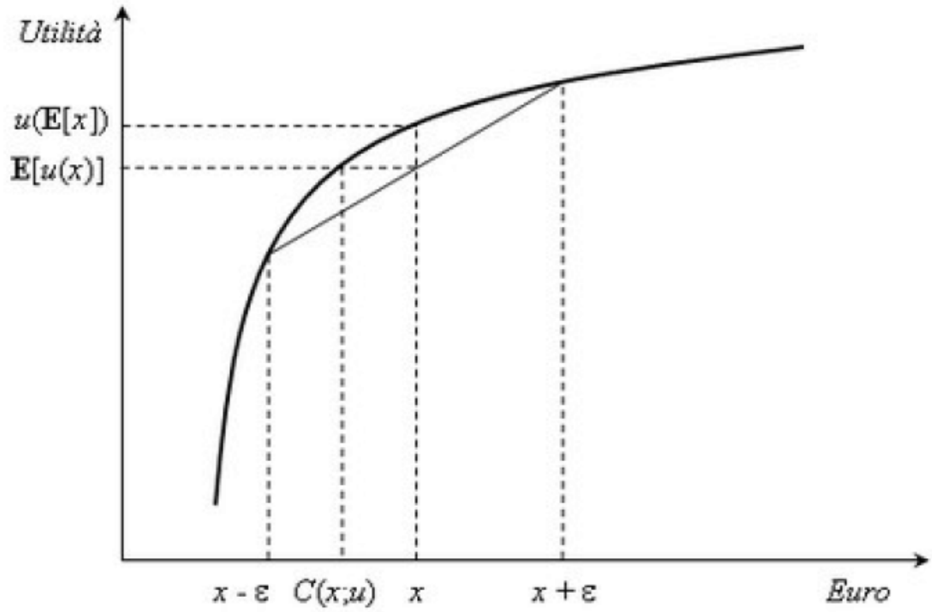

Figure 1a

Defining $\pi_G = \pi$ e $\pi_B = (1 - \pi)$, the figure graphically identify the following values:
x = E(x) = *π(x+ε) + (1-π)(x-ε) = expected (money) value* of the lottery
***u*[*E(x)*]** = *utility of the expected value* of the lottery
***E*[*u(x)*]** = lottery's *expected utility* = *expected utility* of the contingent consumption basket generated by the lottery, according to the von Neumann-Morgenstern utility function:
$Eu = \pi\, u(x + \varepsilon) + (1-\pi)\, u(x - \varepsilon).$

The utility function *u(x)* of a *risk averse* agent is such that ***u*[*E(x)*] > *E*[*u(x)*]**. By definition, a risk-averse agent prefers a sure value ***x*** to a lottery yielding such value only on average (the lottery will pay either (x + ε), or (x - ε), and the agent is averse to not knowing for sure which of the two payments she will receive).

C(x, u) is the *certainty equivalent (CE)* of the lottery, that is, the sure value whose utility is equal to the lottery's expected utility. It is therefore ***u(CE)*** = ***E*[*u(x)*]**. The certainty equivalent ***CE*** is also the maximum price the subject is willing to pay to buy the lottery ticket.

E(x)* − *C(x, u), or in short ***x* – *CE*,** is the *risk premium*, that is, the agent’s risk assessment of participating in the lottery.

In a different context, suppose the agent has the initial endowment $x + \varepsilon$, and is facing the risk that an adverse event, occurring with probability π_B, may reduce her wealth to $x - \varepsilon$. If the incurring damage $D = (x + \varepsilon) - (x - \varepsilon) = 2\varepsilon$, can be totally covered by insurance, she would be willing to spend for comprehensive insurance a maximum sum ***p*** such that:
utility in presence of <u>comprehensive</u> insurance = ***u(x + ε – p)* = *E*[*u(x)*] =** *expected utility in the absence of insurance.*

2. Risk neutrality

Consistently with the above definitions, a risk-neutral individual is indifferent between the random contingent consumption ($x - \varepsilon$, $x + \varepsilon$, π_B, π_G) and the sure consumption ***x* = *E(x)*** corresponding to the expected value of contingent consumption:
$u[E(x)] = u(x) = E[u(x)] = \pi u(x + \varepsilon) + (1 - \pi)u(x - \varepsilon)$.
This occurs if the utility function ***u(x)*** is linear, that is, can be written in the form ***u(x)* = *x***, after an appropriate <u>linear</u> positive transformation. (Notice that a non-linear positive transformation would change the attitude towards risk!) Marginal utility is therefore constant (in our case, ***u’(x)*** = 1). We have:
$E[u(x)] = \pi(x + \varepsilon) + (1 - \pi)(x - \varepsilon) = x = u(x) \equiv u[E(x)]$

In Figure 1b you assume ε = 1, $x - \varepsilon = \overline{c} - 1$ $\quad x + \varepsilon = \overline{c} + 1$ $\quad E(x) = x = \overline{c}$

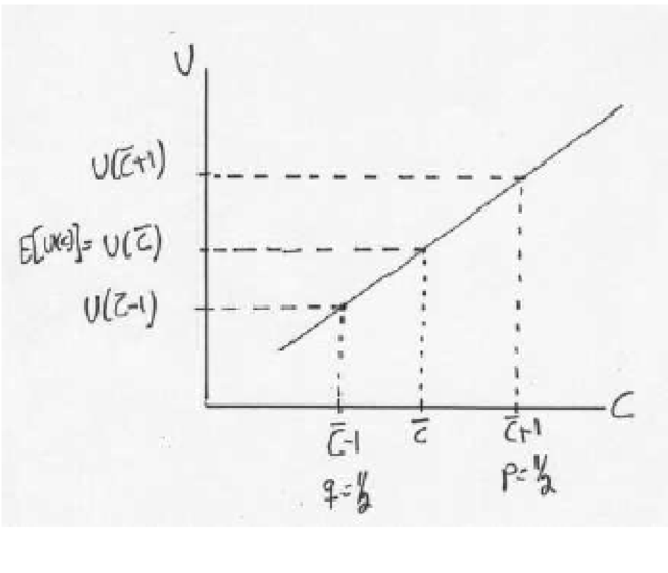

Figure 1.b

3. Neutrality and risk aversion in the space of contingent consumption

Let us now represent risk aversion in the space of contingent consumption. Let c_1 be agent's consumption, contingent on event 1 'sun', and c2 her consumption, contingent on event 2 'bad weather'. Events 1 and 2 occur with probability π_1 e π_2, where $\pi_1 + \pi_2 = 1$.
On average, agent's consumption is the expected consumption $E(c) = \pi_1 c_1 + \pi_2 c_2$.

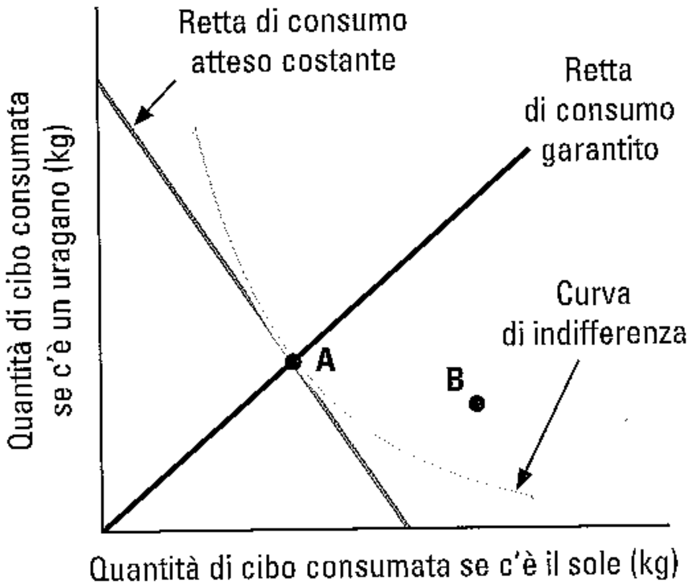

Figure 2

In Figure 2, the downward-sloping *straight line of constant expected consumption* identifies the set of contingent consumption bundles (c_1 , c_2) such that E (c) = constant. The straight line is defined by the equation: $c_2 = E(c) / \pi_1 - (\pi_1 / \pi_2) c_1$.

The slope of the straight line in the plane (c_1 , c_2) is therefore negative: $-\boldsymbol{(\pi_1 / \pi_2)}$.

The 45° line through the origin defines the set of baskets (c_1 , c_2) such that $c_1 = c_2$. On this straight line, consumption is not uncertain (risk is null), that is, consumption is independent of the realization of the events 1 and 2. This justifies the definition *"non-contingent consumption line" (guaranteed consumption)*. The risk taken by the agent increases, as the distance between contingent consumption and guaranteed consumption increases. Contingent consumption bundles further away from the 45° line involve a higher risk.

For a *risk-neutral* agent risk-taking is not costly. As seen previously, her utility function for sure consumption can be expressed in the form $u(c) = c$. It follows that the expected utility of contingent consumption (c_1 , c_2, π_1 , π_2) is expressed by:

expected utility=$E[u(c_1 , c_2, \pi_1 , \pi_2) = \pi_1 u(c_1) + \pi_2 u(c_2) = \pi_1 c_1 + \pi_2 c_2$=*expected consumption* $E(c)$

Proposition: *For a risk-neutral agent, the* <u>*indifference curve*</u> *through a contingent consumption bundle* (c_1,c_2) *coincides with the straight line of expected consumption through* (c_1,c_2).

In Figure 2, because the agent is risk-neutral, the straight line of expected consumption, through any contingent basket, also represents the indifference curve through that basket.
Corollary: *For a risk-neutral agent, the slope of each indifference curve in the contingent consumption plan (c_1 , c_2), that is, the marginal rate of substitution between c_1 and c_2 is:*
$MRS = - \pi_1 u'(c_1) / \pi_2 u'(c_2) = - (\pi_1 / \pi_2)$

Conversely, if a subject is risk averse, the marginal utility of non-contingent consumption is decreasing. This implies that the marginal rate of substitution
$\boldsymbol{MRS = - \pi_1 u'(c_1) / \pi_2 u'(c_2)}$
decreases along an indifference curve as c_1 increases (and therefore c_2 declines). This means that the agent indifference curves are strictly convex in the plane (c_1 , c_2), as shown in Figure 2.

In the same figure we observe that, for a consumer with Von Neuman - Morgestern utility function for contingent consumption, at any basket (like A) such that $c_1 = c_2 = c$ (hence at any basket on the guaranteed consumption line), the *marginal rate of substitution* turns out to be:

$$MRS = - \pi_1 u'(c_1) / \pi_2 u'(c_2) = - \pi_1 u'(c) / \pi_2 u'(c) = - \pi_1 / \pi_2$$

Proposition: *Let A be any basket on the 45° line of the guaranteed consumption. The indifference curve through A is tangent at A to the line of constant expected consumption through A (see figure 3).*

Notice that, if preferences for contingent consumption are described by von Neuman-Morgestern utility functions, then on the 45° line of guaranteed consumption, the MRS of any consumer coincides with the MRS of any other, no matter what their degree of risk aversion may be (in figure 3, Maria and Arnaldo are risk averse, Arnaldo is more risk averse than Maria).

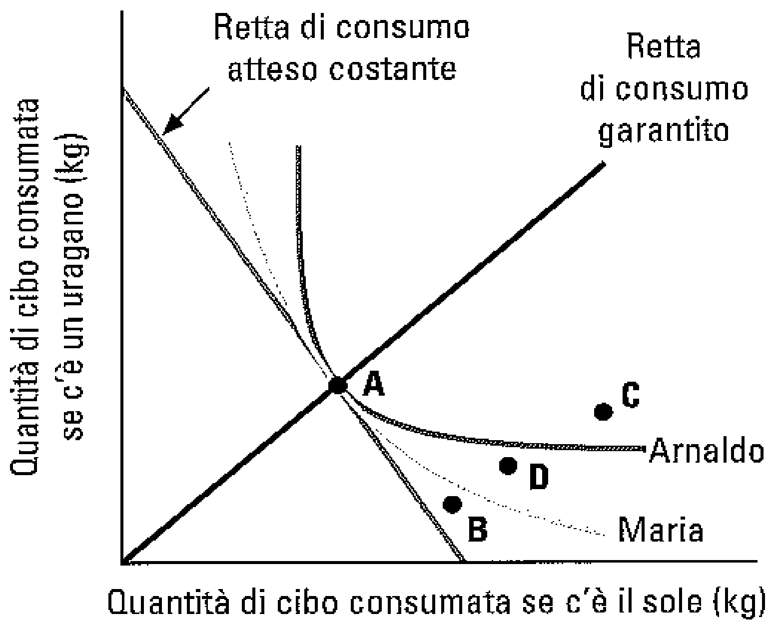

Figure 3

4. Certainty equivalent (CE) and risk premium in the space of contingent consumption

In the space (c_1 , c_2) of contingent consumption, the conditioning events are 1 'sun' and 2 'hurricane'. They are assumed mutually exclusive ($\pi_1 + \pi_2 = 1$). In Figure 4 below, the risky consumption basket B = (b_1 , b_2) and the guaranteed consumption basket A = (a_1 , a_2), with $a_1 = a_2 = a$, lie on the same line of the constant expected consumption. Therefore:

$E(B) = \pi_1 b_1 + \pi_2 b_2 = a$

For a risk-averse consumer we have:
expected utility of B = $EU(B) = \pi_1 u(b_1) + \pi_2 u(b_2) < u(E(B))$ = *utility of the expected value of B* = $u(a)$
As shown in figure 4, B is less preferred than the non-contingent basket A. B is indifferent to a non-contingent basket C = (c_1 , c_2), with $c_1 = c_2 = c$, such that $c < a$. Therefore:

Certainty equivalent of B = $CE(B) = c < a = E(B)$ = *expected value of B.*

A risk-averse subject is willing to exchange B with a non-contingent basket C yielding a lower expected consumption $E(C) = c < a = E(B)$. The agent's degree of risk aversion is measured by the premium (in terms of higher expected consumption), the agent requires to accept the risky bundle B in return for the non-contingent bundle C:

risk premium for B = $E(B) - CE(B) = a - c$

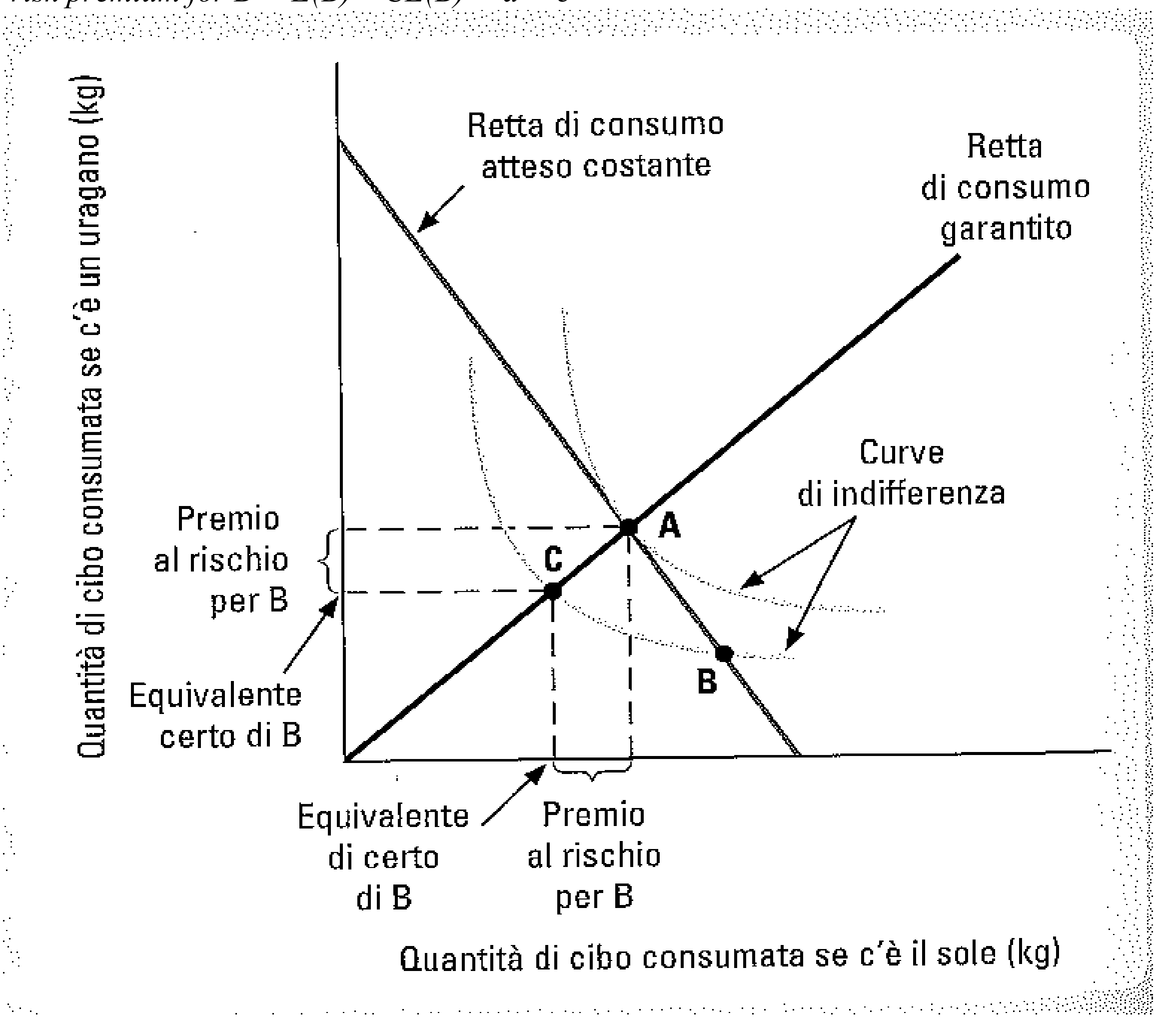

Figure 4