MACROECONOMICS 2018

Discussion Problem set n° 2

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1) Consider the OLG with money seen in class, where the two periods endowments of the consumption good for the representative agent are $e_1 > 0$ and $e_2 > 0$, with the population growing according to the following equation $N_t = (1+n)^t N_0$
Suppose the good is perishable, that is the storage technology is characterized by a return rate $r = -1$.
   a) Write down the budget constraints, for the two periods, and the inter-temporal budget constraint.

Discussion

$c_{1t} = e_1 - m_t \; and \; c_{2t+1} = e_2 + m_t z_t \; where \; m_t = \frac{M_t}{p_t}$ and $z_t = \frac{p_t}{p_{t+1}}$. Hence, the intertemporal budget constraint is given by $c_{2t+1} = e_2 + (e_1 - c_{1t})z_t$

   b) If $m_t = \frac{M_t}{p_t}$, discuss the dynamic stability of monetary stationary state, that is the stability of the real- money market equilibrium equation $m_{t+1} = h(m_t)$, when the utility function of the representative agent $W(c_{1t}, c_{2t+1})$ is
   i) $c_{1t} + \frac{log c_{2t+1}}{1+\theta}$ ii) $c_{1t} + \frac{c_{2t+1}}{1+\theta}$

Discussion

i) $W(c_{1t}, c_{2t+1}) = c_{1t} + \frac{log c_{2t+1}}{1+\theta}$ becomes $W(m_t) = e_1 - m_t + \frac{\log(\; e_2 + \; m_t \, z_t]}{1+\theta}$ which maximized with respect to m_t provides the following FOC

$$-1 + \frac{z_t}{(\; e_2 + \; m_t \, z_t)(1+\theta)} = 0 \;\; (*)$$

From (*) it follows that

$$z_t = \begin{cases} \dfrac{e_2(1+\theta)}{1 - m_t(1+\theta)} & if \; 1 - m_t(1+\theta) > 0 \\ 0 & otherwise \end{cases} \quad (**)$$

Notice that second order conditions are satisfied.
Equilibrium in the monetary market implies

$$M_{t+1}N_{t+1} = H = M_t N_t$$

Hence

$$m_{t+1} = \frac{m_t z_t}{1+n} \quad (***)$$

Replacing (**) into (***) provides

$$m_{t+1} = \begin{cases} \dfrac{m_t e_2(1+\theta)}{(1-m_t(1+\theta))(1+n)} & if\ 1-m_t(1+\theta) > 0 \\ 0 & otherwise \end{cases}$$

Therefore, $m_{t+1} = h(m_t)$ with $h(m_t) = \frac{m_t e_2(1+\theta)}{(1-m_t(1+\theta))(1+n)}$. The stationary states (SS) of the system solve the equation

$$m = \frac{m e_2(1+\theta)}{(1-m(1+\theta))(1+n)}$$

Hence, $m_0 = 0$ is always a SS; moreover, there could also be a monetary SS $m > 0$ equal to

$$m_1 = \frac{1}{(1+\theta)} - \frac{e_2}{(1+n)}$$

if $\frac{1}{(1+\theta)} - \frac{e_2}{(1+n)} > 0$. The first derivative of $h(m_t)$ with respect to m_t is

$$h'(m_t) = \frac{e_2(1+\theta)}{\left(1-m_t(1+\theta)\right)^2(1+n)} > 0$$

which computed in m_0 confirms that the SS $m_1 = \frac{1}{(1+\theta)} - \frac{e_2}{(1+n)} > 0$ if $\frac{1}{(1+\theta)} - \frac{e_2}{(1+n)}$ that is if

$$h'(m_0) = \frac{e_2(1+\theta)}{(1+n)} < 1$$

that is if m_0 is stable, while if m_1 does not exist then m_0 is unstable.

When m_1 exists then $h'(m_1) = \frac{(1+n)}{e_2(1+\theta)} > 1$ and so is unstable.

ii) Discussion of this point is left to the student. Notice that the utility function here is linear

2) Consider the two variable, first order linear difference equation, system

$$x_t = 2x_{t-1} + y_{t-1} + 5$$
$$y_t = 0.3y_{t-1} + 1$$

Find the stationary state of the system and discuss whether or not is a saddle point.

Discussion

The system could be written in matrix term as

$$u_t = Au_{t-1} + b$$

Where $u_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}$, $A = \begin{bmatrix} 2 & 1 \\ 0 & 0.3 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. The stationary state is $u_s = \begin{bmatrix} x_s = -45/7 \\ y_s = \ \ 10/7 \end{bmatrix}$

Note that u_s solves the equation $u_s = Au_s + b$ hence $u_s = (I-A)^{-1}b$ where $(I-A)^{-1}$

$$(I-A)^{-1} = \begin{bmatrix} -1 & -10/7 \\ 0 & 10/7 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 0{,}3$. Therefore, u_s is a saddle point and the unique path converging to it the saddle path. But what is the saddle path in this variation of the linear model, with the vector b adding up to Au_{t-1}?

This is how to proceed. The model can be written as $u_t = A(u_{t-1} + u_s - u_s) + b$. Define $z_t = u_t - u_s$ and so

$$u_t = Az_{t-1} + Au_s + b$$

But

$$Au_s + b = A(I-A)^{-1}b + b = (I + A(I-A)^{-1})b = ((I-A)(I-A)^{-1} + (I-A)^{-1})b$$

hence

$$Au_s + b = ((I-A) + A)(I-A)^{-1}b = u_s$$

Therefore

$$u_t = Az_{t-1} + Au_s + b$$

can be written as

$$z_t = Az_{t-1}$$

As stationary state z_s the above system has the origin $z_s = 0$. Therefore, the saddle path of z_t is the eigenvector $v_2 = \begin{bmatrix} v_{12} = & -17x_2/10 \\ v_{22} = & R \end{bmatrix}$ of A associated to $\lambda_2 = 0{,}3$. As a consequence, if $z_0 = u_0 - u_s$ must be located on the vector v_2 to converge to $z_s = 0$ it follows that $u_0 = z_0 + u_s$, that is the u_t must be initially located on the vector v_2 to which the stationary state is added. This means that u_0 must be located over the parallel to v_2 crossing u_s, as we saw in class.