RETURNS OF SCALE (1 OUTPUT)

1) CONSTANT RETURNS OF SCALE

CRS

a) $y \in Y \Rightarrow ty \in Y \quad \forall t > 0$

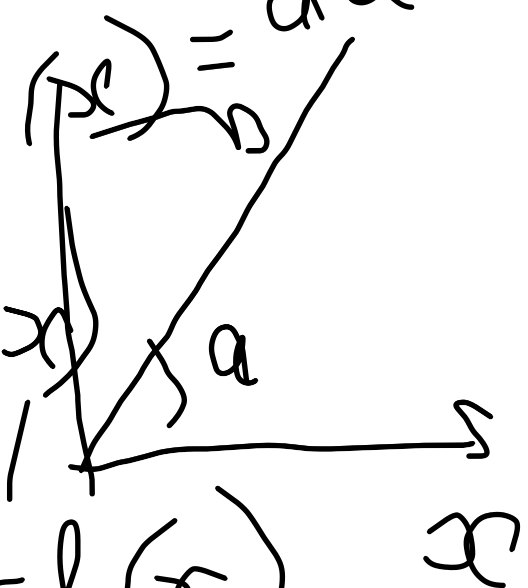
b) $x \in V(z) \Rightarrow tx \in V(tz) \quad \forall t > 0$

c) $f(tx) = tf(x) \quad \forall t > 0$

EX 1 OUTPUT 1 INPUT $f(x) = ax$

$f(x) = ax \quad a > 0$

$f(tx) = a(tx) = t(ax) = tf(x)$



EX $f(x)$ $\quad x = (x_1, x_2)$

$$f(x_1, x_2) = a x_1 + b x_2$$

$$\boxed{f(tx)} = \boxed{f(tx_1, tx_2)} = a\boxed{(tx_1)} + b\boxed{(tx_2)} =$$

$$= t(a x_1 + b x_2) = t f(x_1, x_2) = t f(x)$$

$$tx = (tx_1, tx_2, \ldots, tx_n)$$

EX $f(x)$ $\quad x = (x_1, x_2)$

$$f(x) = f(x_1, x_2) = x_1^{a} x_2^{b} \quad \text{(COBB-DOUGLAS)}$$

$$\boxed{f(tx)} = (tx_1)^a (tx_2)^b = t^{a+b} x_1^a x_2^b = \boxed{t^{a+b} f(x)}$$

if $a + b = 1$ $\Rightarrow$ CRS

2) INCREASING RETURNS OF SCALE

$$f(tx) > t f(x) \qquad \boxed{\forall t > 1}$$

IRS

$t = 2$

$\boxed{f(tx)} \overset{?}{>} 2 f(x)$

EX $t = 2$

$f(x) = 3 \Rightarrow f(tx) = f(2x) = 7 > 2 f(x)$

$2 \times 3 = 6$

3) DECREASING RETURNS OF SCALE

$$f(tx) < t f(x) \qquad \forall t > 1$$

EX

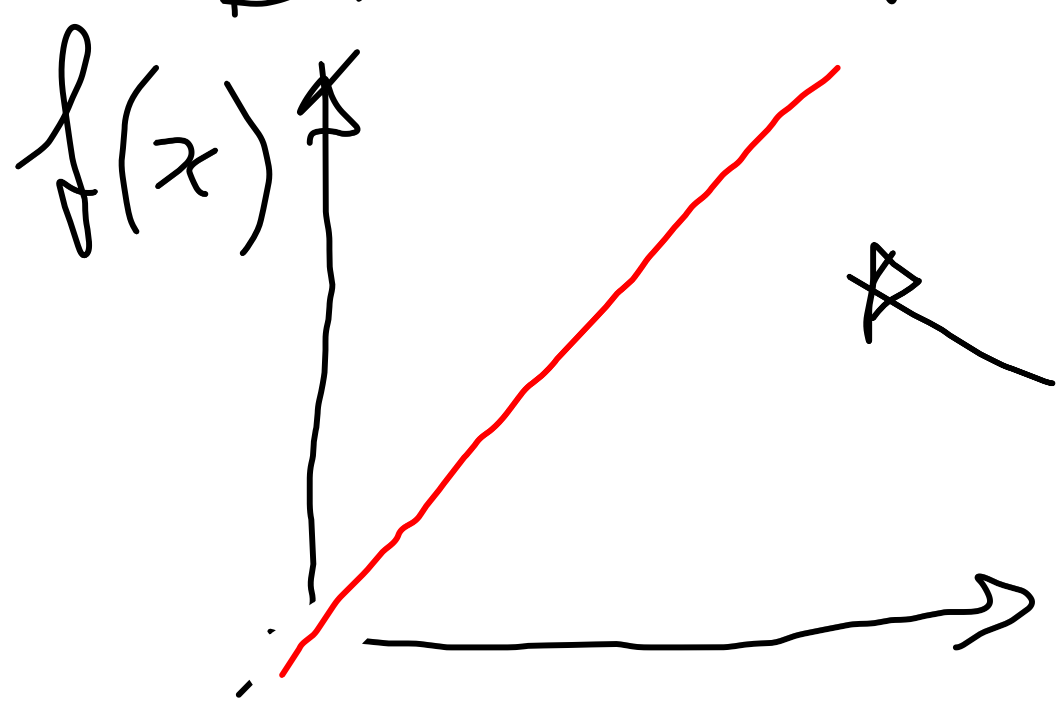
$f(x) = 3 \qquad t = 2$

$f(2x) = f(tx) = 5 < t\,f(x) = 2 \cdot 3 = 6$

$\Downarrow$

DRS

EX $\qquad$ 1 OUTPUT $\qquad$ 1 INPUT

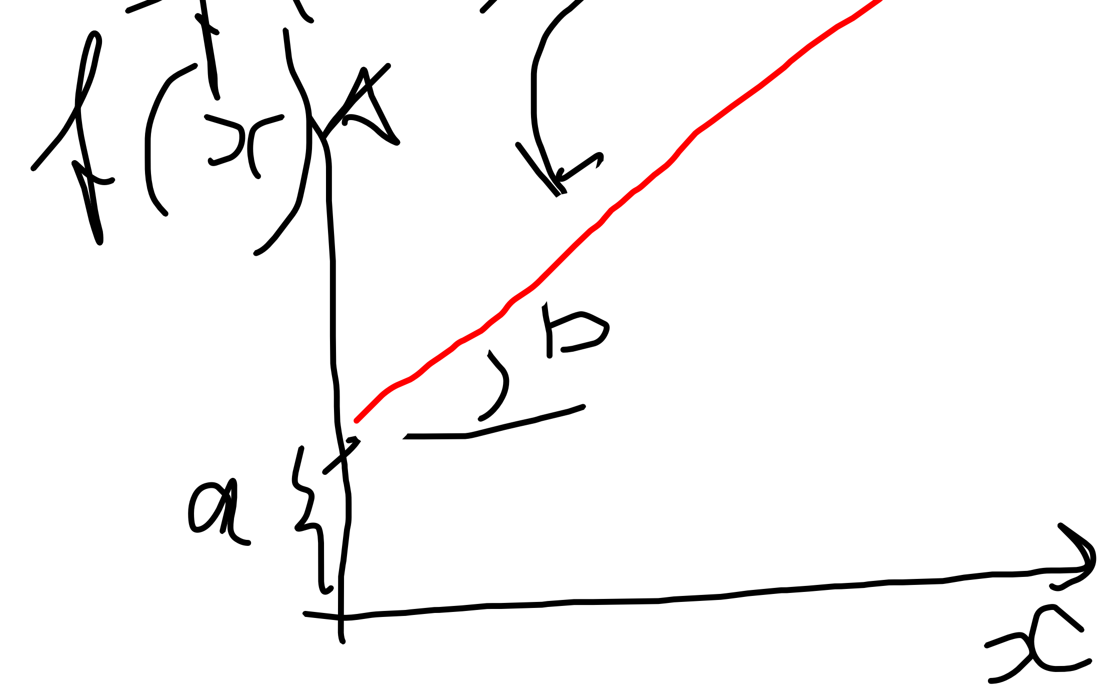


$f(x) = ax$

$\Downarrow$

CRS

$f(x) = a + bx$



$$f(x) = a + bx$$

$$\boxed{f(tx)} = \boxed{a + b(tx)} = \quad \forall t > 0 \text{ and } t \neq 1$$

$$= \boxed{a + t(bx)} \neq \boxed{t\, f(x)}$$

IN FACT

$$t\, f(x) = t\,\overbrace{\left[a + bx\right]}^{f(x)} =$$

$$= \boxed{ta + t(bx)}$$

SO, WHAT KIND OF RETURNS OF SCALE WOULD THE LINEAR PRODUCT FUNCTION HAVE?

$$\boxed{f(x) = a + bx} \qquad b > 0$$

IF $t > 1$

$$f(tx) = \boxed{a + t(bx) \lessgtr} \; \boxed{ta + t(bx)} = t f(x)?$$

$$\Rightarrow \quad a + t(bx) < t[a + bx] \Rightarrow f(tx) < t f(x)$$

DECREASING RETURNS OF SCALE

EX

$$f(x_1, x_2) = x_1^a \, x_2^b \qquad t > 1$$

$$f(tx) = t^{a+b} \qquad f(x) \gtreqless t\, f(x)$$

IF $a+b > 1$

$$\Longrightarrow f(tx) > t f(x) \Rightarrow IRS$$

IF $a+b < 1$

$$\Longrightarrow f(tx) < t f(x) \Rightarrow DRS$$

PROFIT FUNCTION (COMPETITIVE MARKET, WORLD)

$$\Pi = \underset{\text{REVENUES}}{R} - \underset{\text{COSTS}}{C}$$

$$\begin{cases} y = (2, -1, 3, -4) \\ p = (1, 3, 2, 5) \end{cases} \qquad n = 4$$

$$\Pi = \boxed{\underbrace{(2 \times 1 + 3 \times 2)}_{R} - (\underbrace{1 \times 3 + 4 \times 5}_{C})} = py$$

$$\underset{1\times 1}{py} = \overset{p}{\underset{1\times 4}{(1, 3, 2, 5)}} \underset{4\times 1}{\overset{y}{\begin{pmatrix} 2 \\ -1 \\ 3 \\ -4 \end{pmatrix}}} = 1\times 2 + 3(-1) + 2(3) + 5(-4) =$$

$$\underbrace{(1\times 2 + 2\times 3)}_{R} - \underbrace{(3\times 1 + 4\times 5)}_{C}$$

$$\Longrightarrow \boxed{\underset{1\times 1}{\Pi} = \underset{1\times m}{p}\,\underset{m\times 1}{y}}$$

WITH PERFECT COMPETITION (P GIVEN)

$$\boxed{\max_y \underset{\text{GIVEN}}{p}\,y \quad \text{SUCH THAT } y \in Y}$$

$\Pi(p) = p\,y(p)$ PROFIT FUNCTION?

$$\max_y \boxed{py} \quad \text{SUCH THAT } y \in Y$$

PROFIT EXPRESSION

$\Longrightarrow y(p)$ SOLUTION $\Longrightarrow$ $\boxed{p\,y(p) = \Pi(p)}$

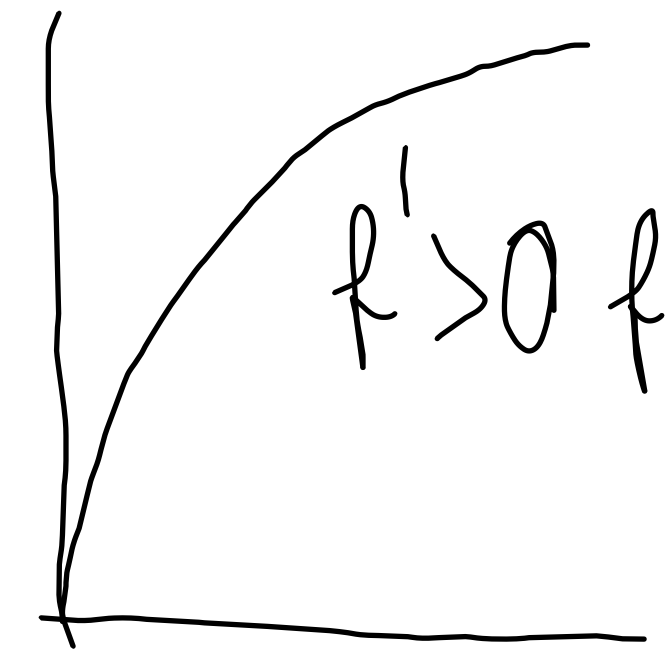
MAX PROFIT

PROFIT FUNCTION = MAX PROFIT THAT CAN BE OBTAINED BY THE FIRM (FUNCTION OF P) W

EX 1 OUTPUT 1 INPUT

$z = f(x)$ $\qquad x$

p $\qquad w$



$$\max_{z,x} \Pi = pz - wx = \boxed{\underbrace{pf(x)}_{R} - \underbrace{wx}_{C}}$$

EX $f(x) = \sqrt{x} = x^{1/2}$

$y = (z, x)$

$\boxed{f(x)}$

$$\Rightarrow \max_x \; pf(x) - wx$$

$\Rightarrow$ FOC $\quad pf'(x) - w = 0$

SOC $\quad pf''(x) < 0$ YES IF $f'' < 0$

EX

$$f(x) = \sqrt{x} \Rightarrow f' = \frac{x^{-1/2}}{2} > 0 \Rightarrow f'' = -\frac{1}{4} x^{-3/2} < 0$$

$$\max_x \Pi(x) = p\sqrt{x} - wx$$

FOC

$$\Rightarrow \frac{p}{2\sqrt{x}} - w = 0 \Rightarrow \left(\frac{p}{2w}\right)^2 = x(p,w)$$

$$f(x) = \sqrt{x} = \sqrt{\left(\frac{p}{2w}\right)^2} = \frac{p}{2w} = z(p,w)$$

$$\Rightarrow y = (z, -x) \Rightarrow \boxed{y(p,w)} = \left(z(p,w), x(p,w)\right)$$

$$\frac{p}{2w} \qquad \left(\frac{p}{2w}\right)^2$$

$$\Rightarrow \Pi = p\sqrt{x} - wx$$

$$\Pi = p\overbrace{\sqrt{x}}^{z} - wx$$

REPLASE $z = \dfrac{p}{2w}$ $\quad x = \left(\dfrac{p}{2w}\right)^2$

$$\to \Pi = p\left(\frac{p}{2w}\right) - w\left(\frac{p}{2w}\right)^2 =$$

$$= \frac{p^2}{2w} - \frac{p^2}{4w} = \boxed{\frac{p^2}{4w}} = \Pi(p, w)$$

PROFIT FUNCTION

EX

$f(x) = ax \qquad a > 0$

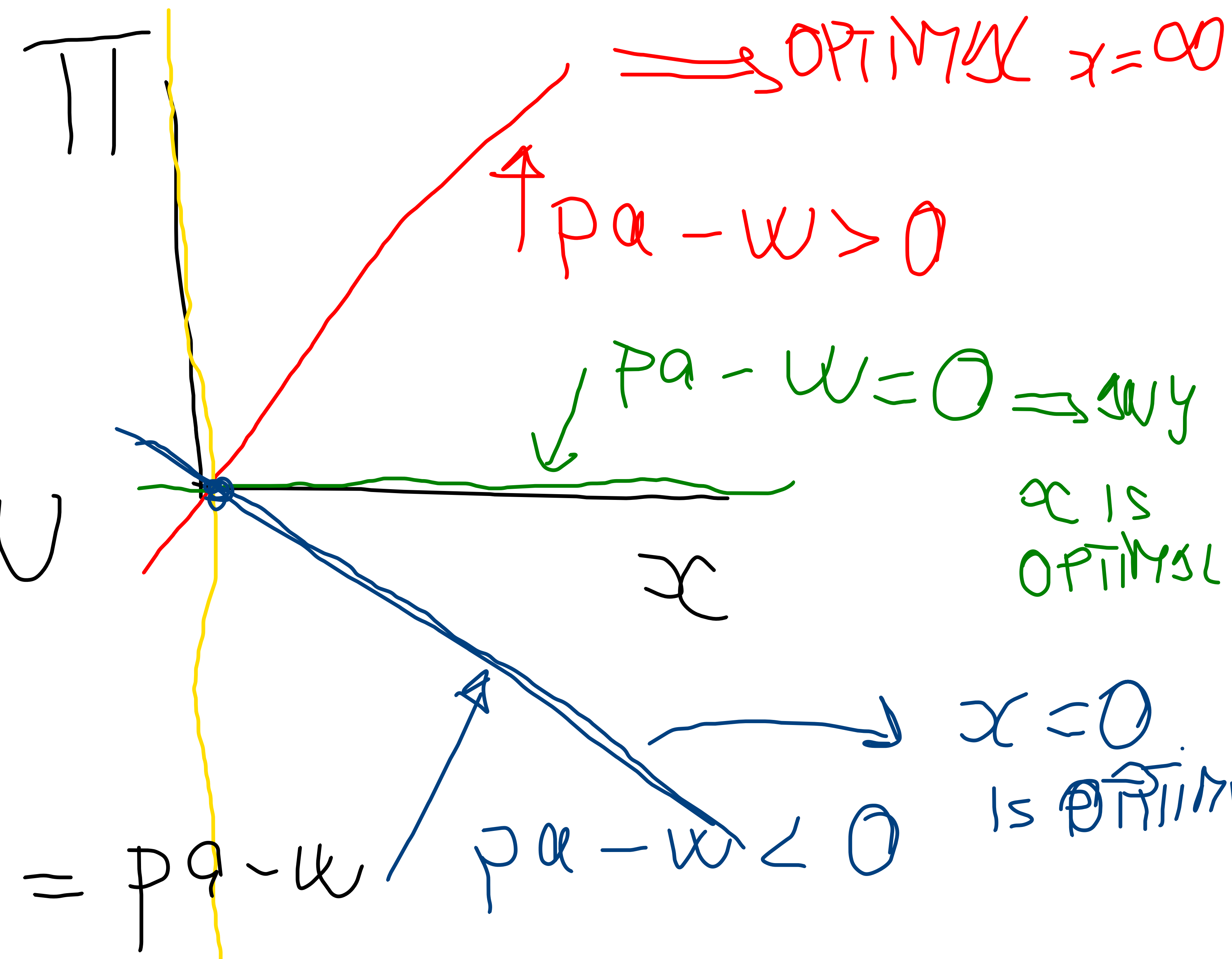
1 OUT

y INP

$$\max_{x} \Pi = p(\underset{z}{ax}) - wx = x[pa - w]$$

WHAT IS THE SOLUTION OF THE PROBLEM?

$$\frac{d\Pi}{dx} = \frac{d[x(pa-w)]}{dx} = pa - w$$

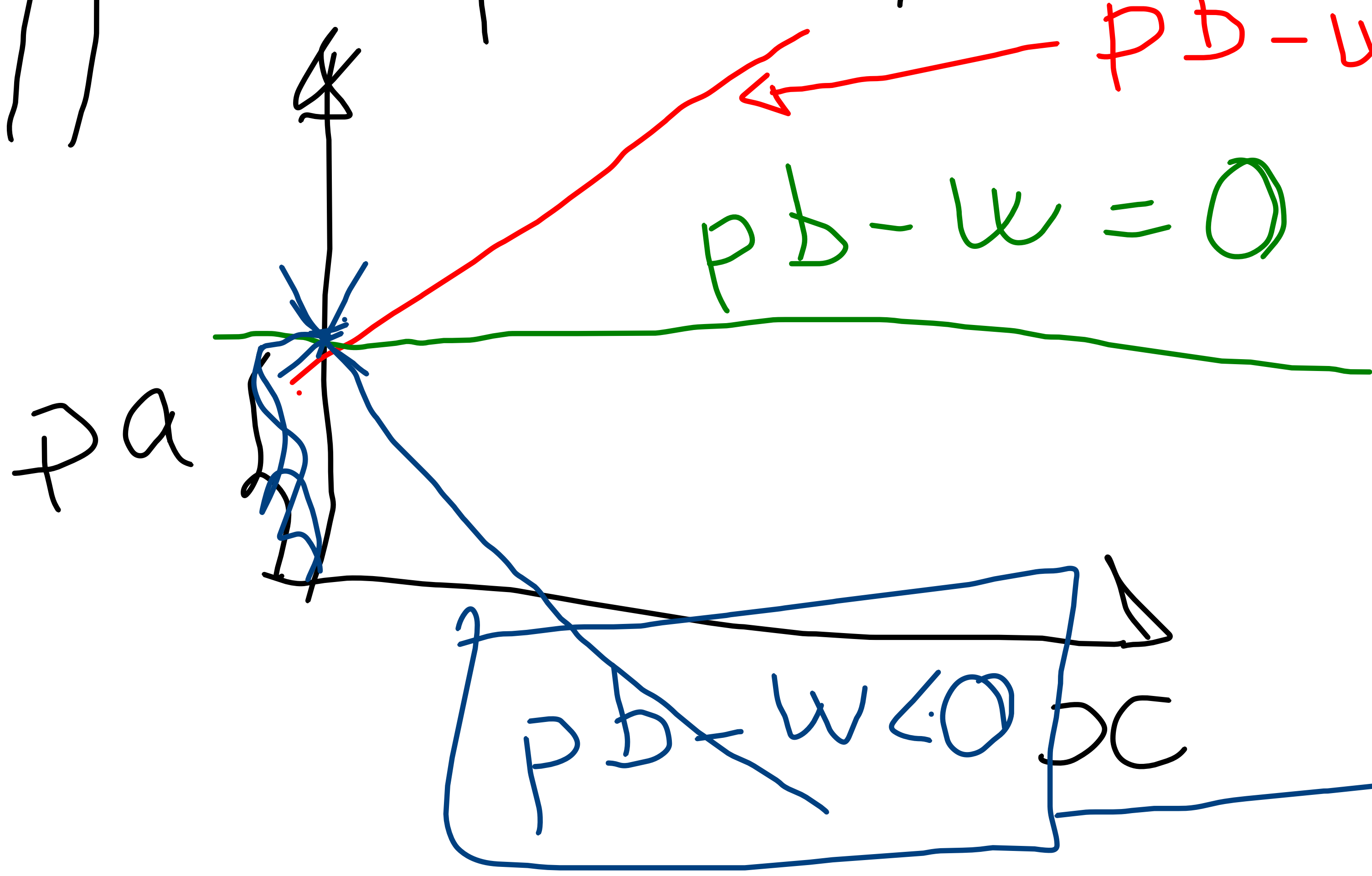


EX

$$\boxed{f(x) = a + \cancel{bx} = z} \qquad a > 0$$

$$\Pi = p[a + bx] - wx =$$

$$\Pi = pa + pbx - wx = \boxed{pa} + x[pb - w]$$



$\rightarrow$ OPTIMAL $x = 0$

NEXT TIME Thursday 7th

PROPERTIES OF $\Pi(p)$

PROFIT FUNCTION

$$\pi(p) \Rightarrow \boxed{\max_y \; py \quad \text{st } y \in Y}$$

PROFIT FUNCTION

$$y(p)$$

$$\Downarrow$$

$$\boxed{\pi(p) = p\,y(p)}$$

ANALOGUE OF INDIRECT UTILITY FUNCTION

EX

$\uparrow$ OUTPUT

$\uparrow$ INPUT

PRICE OF INPUT (LABOUR)

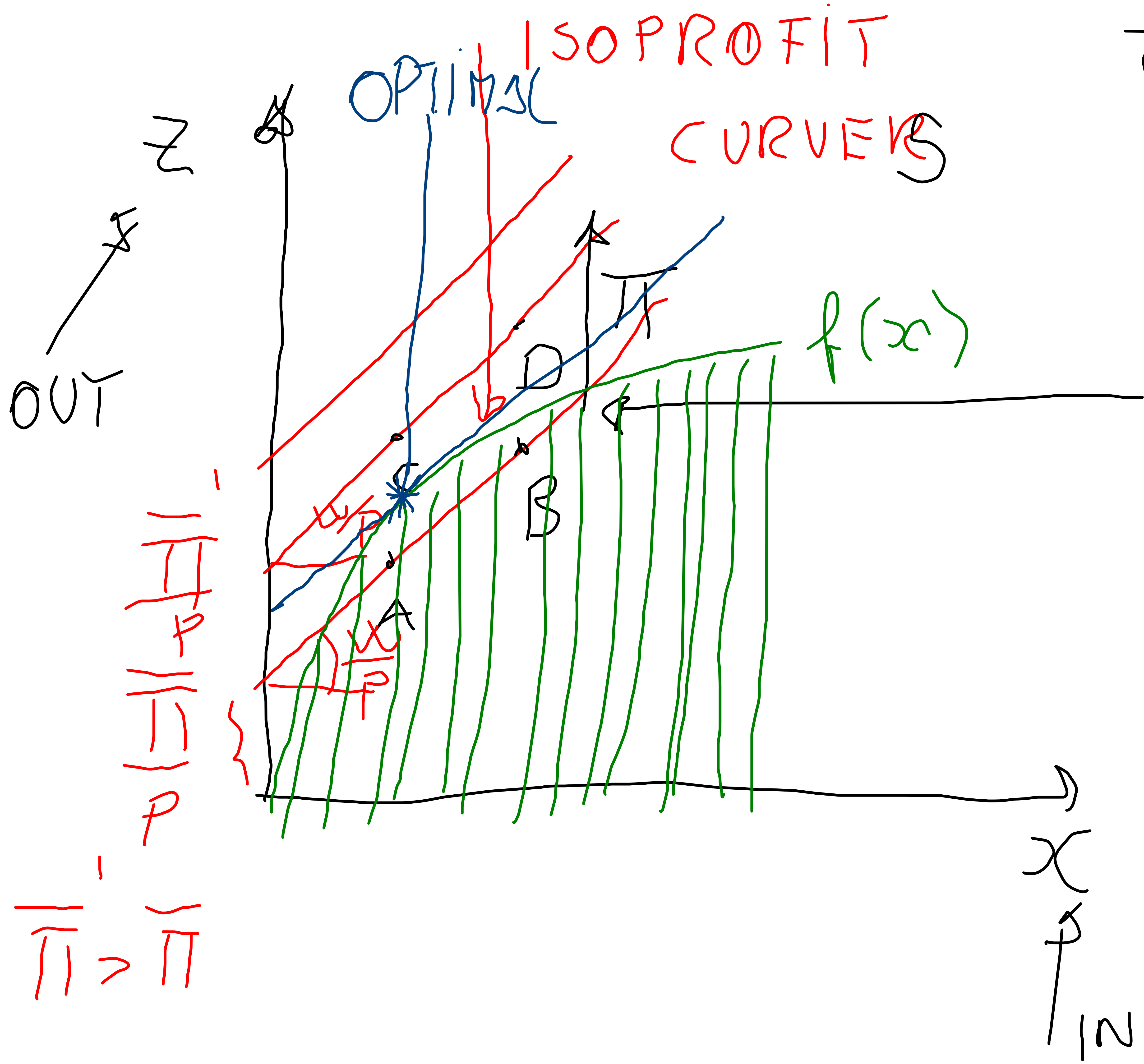
$$\pi(x) = p f(x) - w x \implies \pi'(x) = p f'(x) - w = 0$$

PRICE OF OUT.

$$\implies \boxed{f'(x) = \frac{w}{p}}$$

REAL WAGE

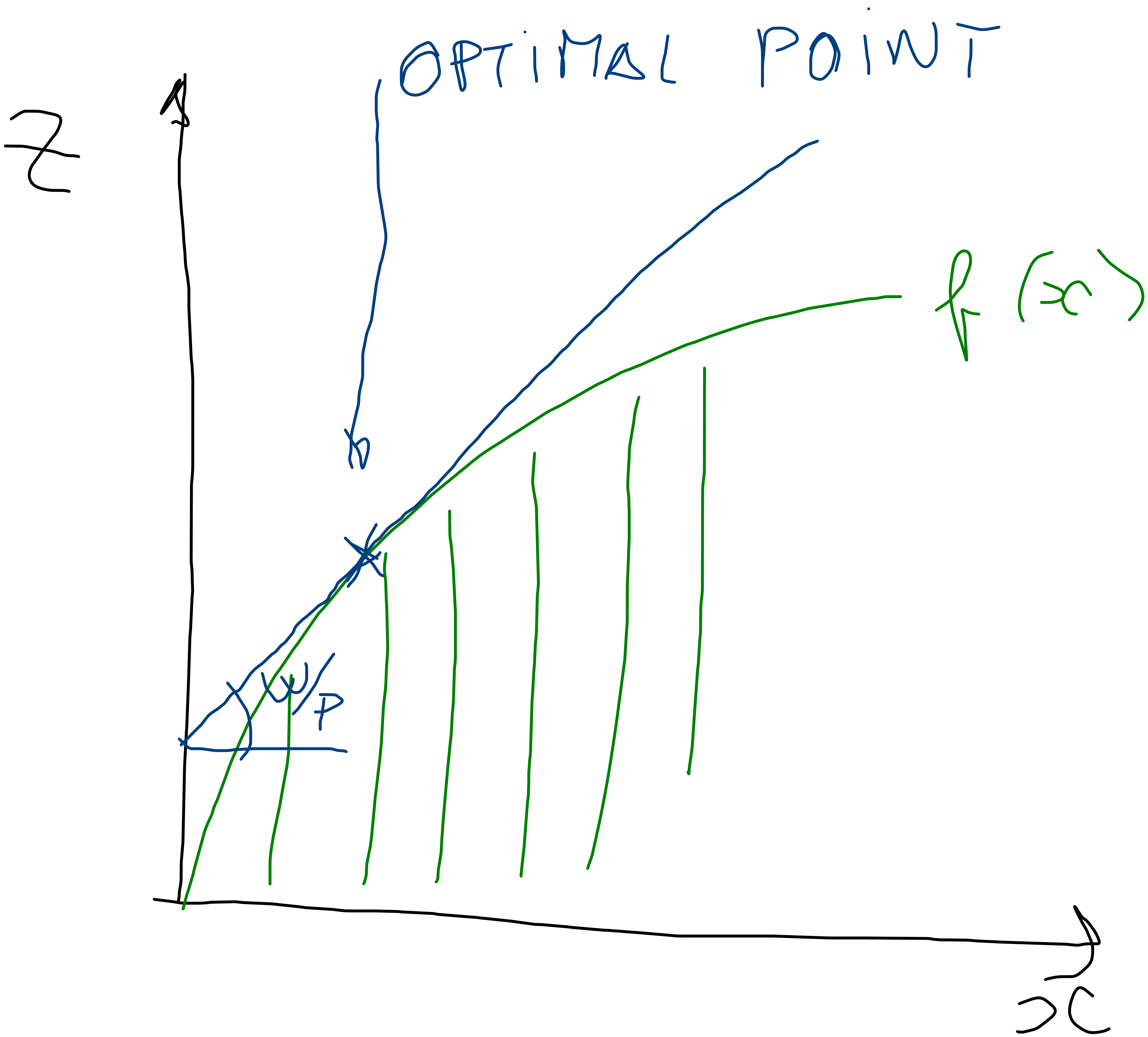
MARG PRODUCTIVITY



$$\pi = pz - wx$$

$$\Downarrow$$

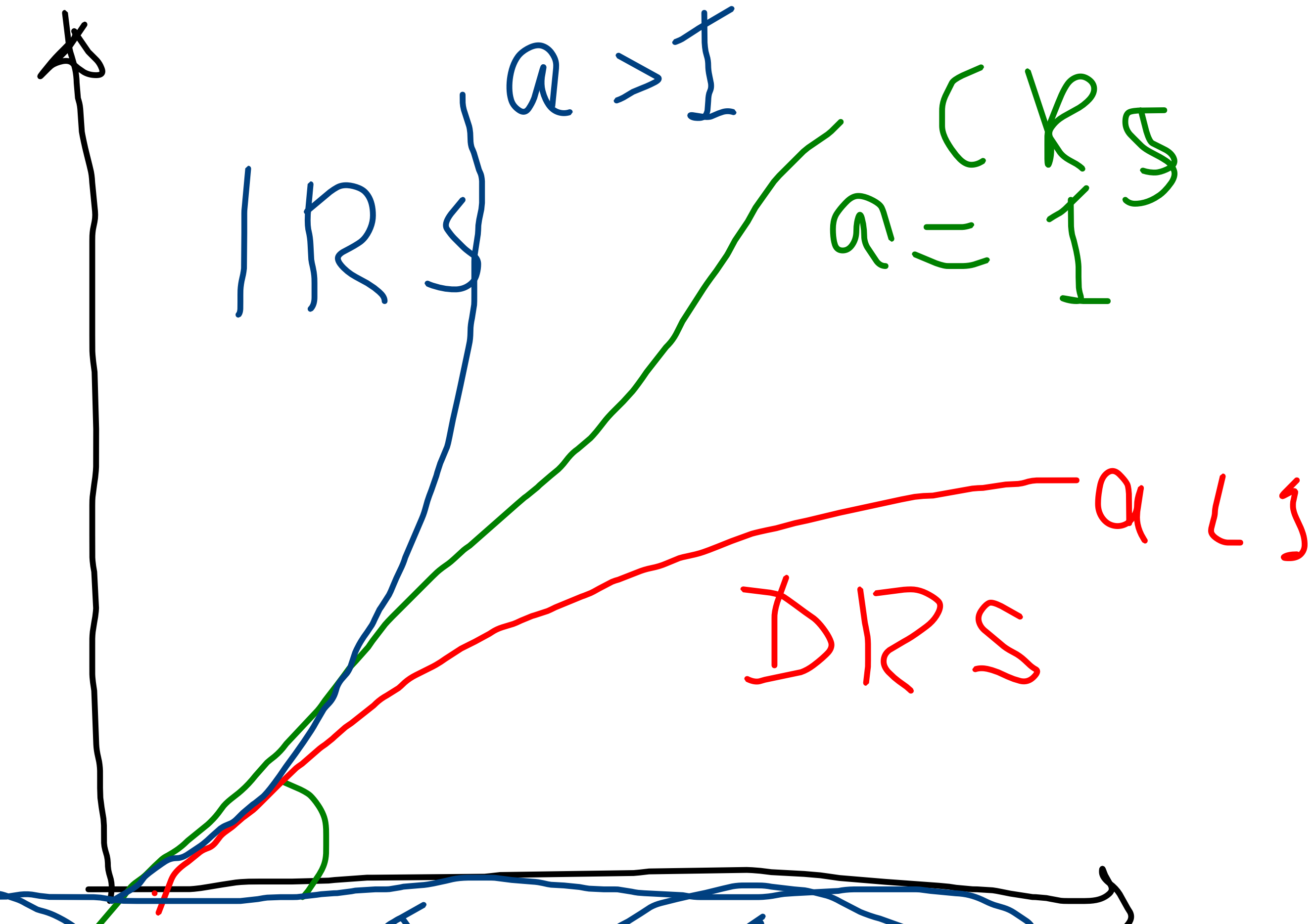
$$z = \frac{\pi}{p} + \frac{w}{p} x$$



$$\boxed{f'(x) = \frac{w}{p}}$$

EX

$$f(x) = x^{a} \qquad a > 0$$



$$y(p,w) = f(x(p,w))$$

$$\max_{z,x} \Pi = pz - wx$$

OPTIMAL $\boxed{x(p,w) = \left(\frac{p}{w}\right)^{\frac{1}{1-a}} a^{\frac{1}{1-a}}}$

$$\Pi = p f(x) - wx = p x^{a} - wx$$

$$\Longrightarrow \Pi' = p a x^{(a-1)} - w = 0 \Longrightarrow x = \left(\frac{w}{pa}\right)^{\frac{1}{a-1}}$$

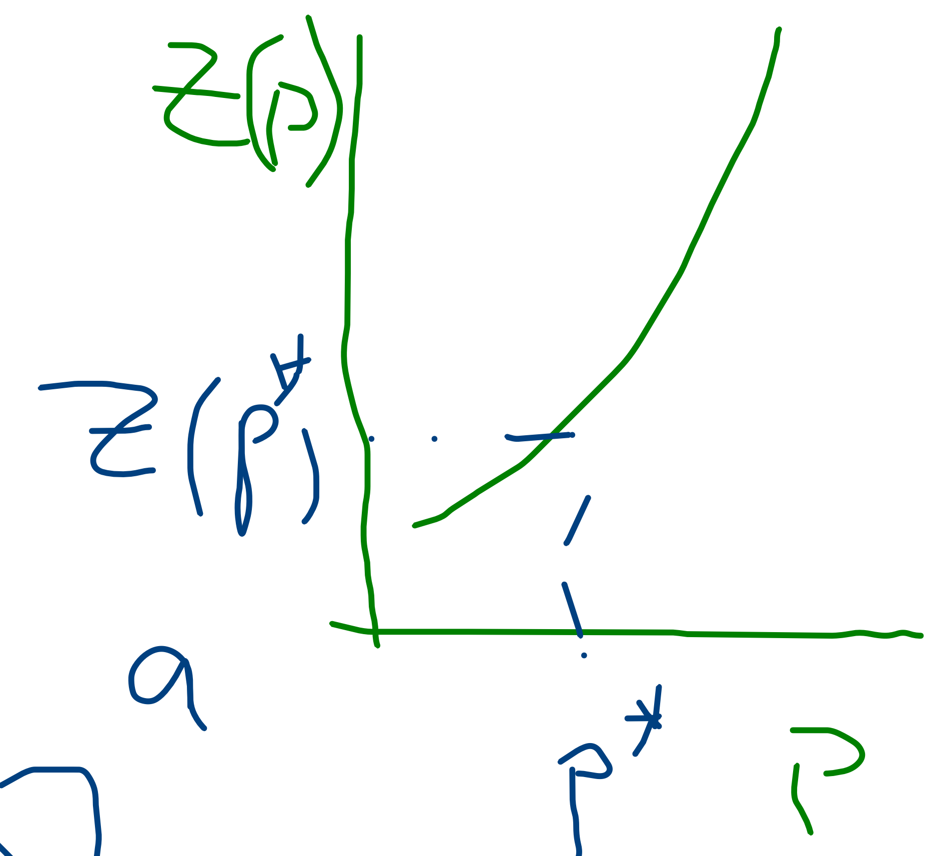
$$\Downarrow$$

$$\boxed{x = \left(\frac{pa}{w}\right)^{\frac{1}{1-a}}}$$

$$\Pi'' = p a (a-1) x^{a-2} < 0 \Longrightarrow a < 1$$

$$x(p,w) = \left(\frac{p}{w}\right)^{\frac{1}{1-a}} a^{\frac{1}{1-a}}$$

OPT. INPUT



$\downarrow$

OPT OUT ?

$$f(x) = x^a \Rightarrow f(x(p,w)) = [x(p,w)]^a =$$

$$\boxed{= \left(\frac{p}{w}\right)^{\frac{a}{1-a}} a^{\frac{a}{1-a}}}$$

OPT. OUT

SUPPLY FUNCTION

$$z(p,w) = \left(\frac{p}{w}\right)^{\frac{a}{1-a}} a^{\frac{a}{1-a}}$$

SUPPLY FUNCT $\left(\frac{P}{w}\right)^{\frac{a}{1-a}} \left(a\right)^{\frac{a}{1-a}}$

$0 < a = 1/2 < 1$

$\Longrightarrow Z(P,w) = \frac{P}{2w} \Longrightarrow$

$\Pi(p,w) = PZ - wx = p\left[\left(\frac{Pa}{w}\right)^{\frac{a}{1-a}}\right] - w\left[\left(\frac{Pa}{w}\right)^{\frac{1}{1-a}}\right] P$

$\frac{d\Pi(p,w)}{dp} \gtreqless 0\,?$ $\quad \frac{d\Pi(p,w)}{dw} \gtreqless 0\,?$

PROFIT FUNCT $\quad$ SUPPLY FUNCT $Z(P,w)$ $\quad$ $x(P,w)$ INPUT DEMAND FUNCT

PROPERTIES OF PROFIT FUNCTION

$$\underset{1\times m}{\Pi(p)} = \underset{1\times m}{p}\,\underset{m\times 1}{y(p)} \iff \max_y \; py \quad \text{s.t. } y \in Y$$

1) $\Pi(p)$ IS NON DECREASING IN THE OUTPUTS PRICES AND NON INCREASING IN THE INPUTS PRICES

PROOF

$p' \to p_i' \geq p_i$ WHERE i IS AN OUTPUT

$\to p_j' \leq p_j$ WHERE j IS AN INPUT

$\to \Pi(p') \geq \Pi(p)$ INDEED $\boxed{\Pi(p') = p'y' \geq p'y}$

WHERE y' IS PROFIT MAX AT p' AND y IS PROFIT MAX AT p

$$\Pi(p') = p'y' \geq p'y \geq py = \Pi(p)$$

$p'_i \geq p_i$ i OUTPUTS IN y

$p'_j \leq p_j$ j INPUTS IN y

EX

$$y = (2 \quad 3 \quad -4 \quad -5)$$

$$p = (1 \quad 2 \quad 3 \quad 4)$$

$$\qquad\ \ \wedge \quad \wedge \quad \vee \quad \vee$$

$$p' = (\underbrace{2 \quad 3}_{\text{OUT}} \quad \underbrace{2 \quad 3}_{\text{INP}})$$

$$\Longrightarrow \boxed{yp} < \boxed{yp'}$$

2) $\Pi(p)$ is <u>CONVEX</u> in p

NAMELY

IF p, p' ARE TWO PRICE VECTORS DEFINE

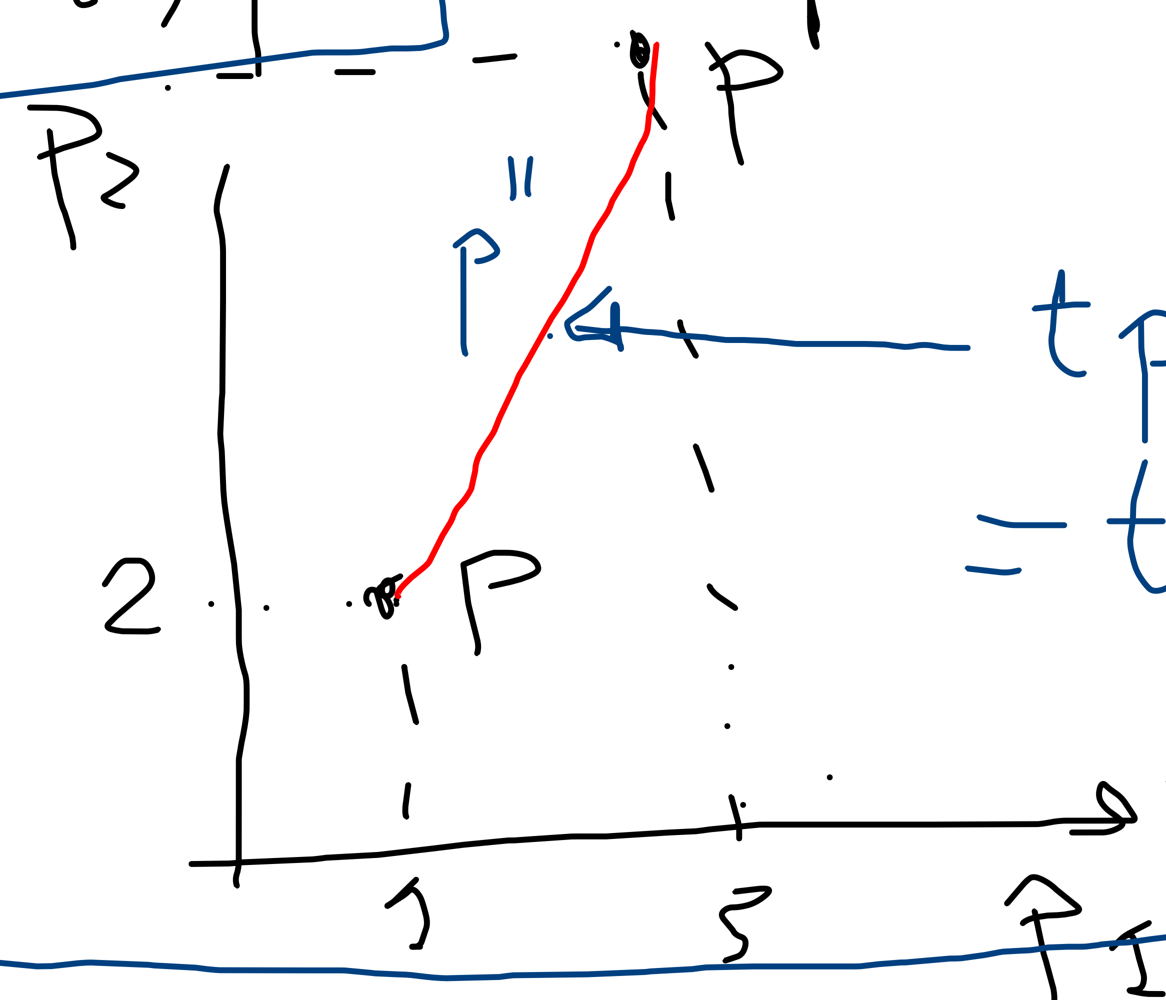
$$\boxed{p'' = tp + (1-t)p'} \qquad 0 \le t \le 1$$

<u>EX</u>

$p_1 \quad p_2$

$p = (1, 2)$

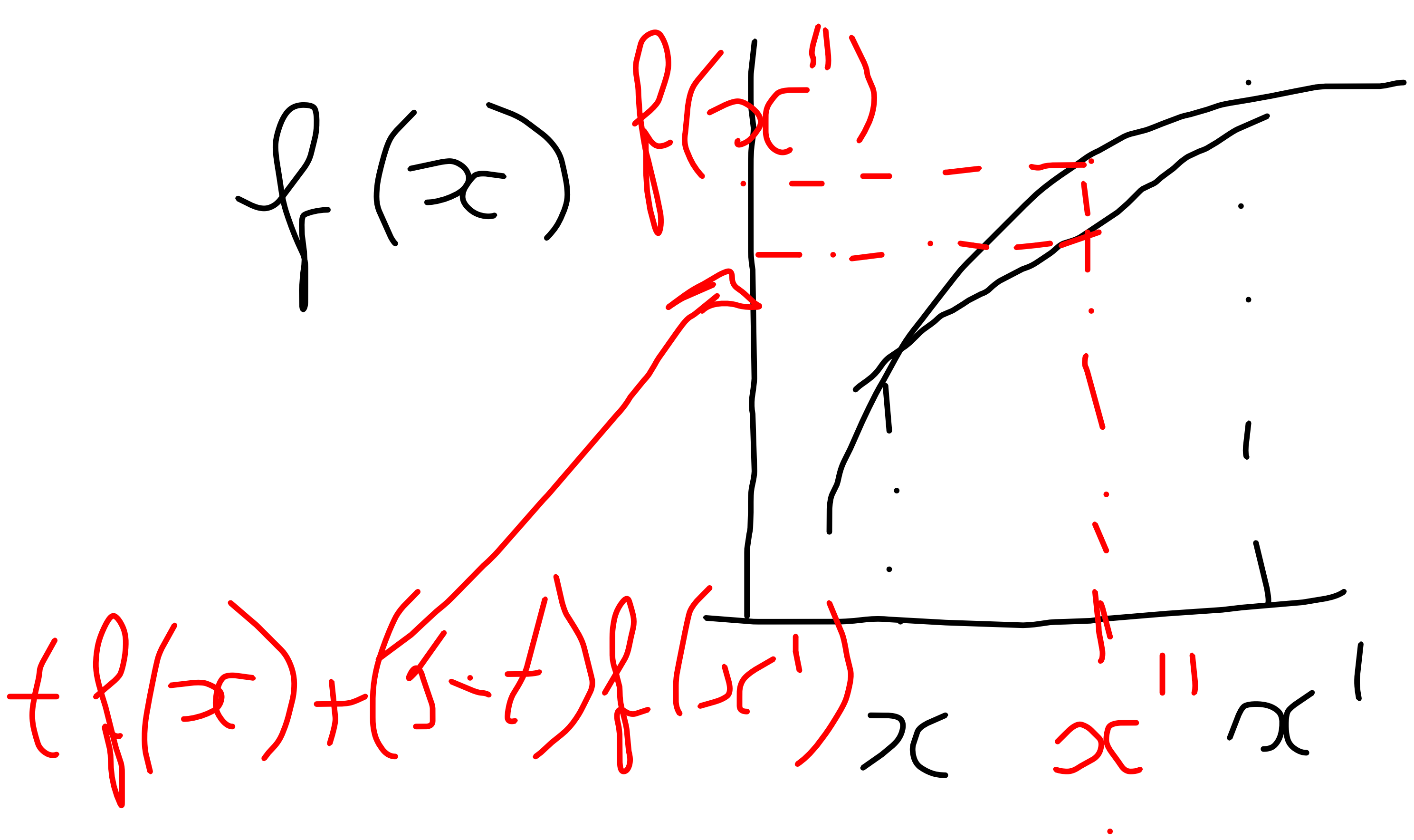
$p' = (3, 4)$



$$tp + (1-t)p' = t(1,2) + (1-t)(3,4) = (t \cdot 1 + (1-t)3,\; t \cdot 2 + (1-t)4)$$

$$\rightarrow \boxed{\Pi(p'') \le t\,\Pi(p) + (1-t)\,\Pi(p')}$$

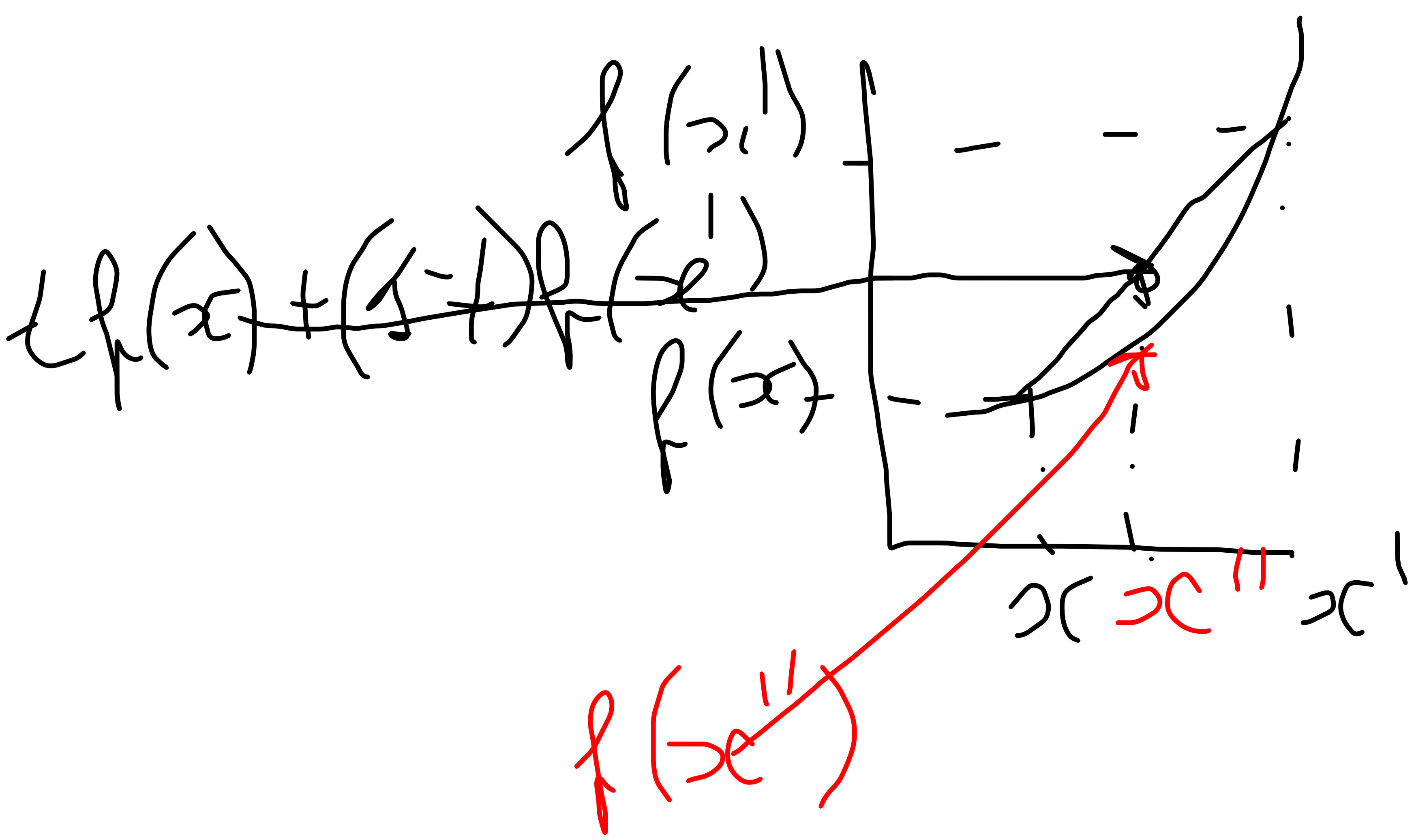
REMEMBER



$$f(x'') \geq t f(x) + (1-t) f(x')$$

$$\Downarrow$$

CONCAVITY



$$f(x'') \leq t f(x) + (1-t) f(x')$$

$$\Downarrow$$

CONVEXITY

PROOF

TAKE $p'' = tp + (1-t)p'$ AND SUPPOSE y IS PROFIT MAX AT p, y' IS PROFIT MAX AT p' AND y'' IS PROFIT MAX AT p''

$$p''y'' = [tp + (1-t)p']y'' = tpy'' + (1-t)p'y'' \leq tpy + (1-t)p'y'$$

$$\boxed{\pi(p'') \leq t\pi(p) + (1-t)\pi(p')}$$ Q.E.D.

3) $\pi(p)$ DEGREE 1 HOMOGENEOUS $t > 0$

$$\pi(tp) = t\pi(p)$$