UNIVERSITA' DEGLI STUDI DI SIENA
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Intermediate Test Quantitative Methods for Economic Applications - Mathematics (25/11/21)

1) Given the complex number $z = \frac{2}{1+i}$. Calculate its cubic roots.

By rationalisation we get $z = \frac{2}{1+i} = \frac{2}{1+i} \cdot \frac{1-i}{1-i} = \frac{2(1-i)}{1-i^2} = \frac{2(1-i)}{1+1} = \frac{\cancel{2}(1-i)}{\cancel{2}} = (1-i)$. The modul of z is $\rho_z = \sqrt{1^2 + (-1)^2} = \sqrt{2}$, hence z can be rewritten as $z = \sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = \sqrt{2}\left(cos\frac{7}{4}\pi + i\,sin\frac{7}{4}\pi\right)$. The argument of z is $\frac{7}{4}\pi$. The three cubic roots of z can be calculated by the formula

$$\sqrt[3]{z} = \sqrt[3]{\sqrt{2}}\left(cos\left(\frac{7\pi/4 + 2k\pi}{3}\right) + i\,sin\left(\frac{7\pi/4 + 2k\pi}{3}\right)\right) =$$

$$\sqrt[6]{2}\left(cos\left(\frac{7}{12}\pi + \frac{2\pi}{3}k\right) + i\,sin\left(\frac{7}{12}\pi + \frac{2\pi}{3}k\right)\right)$$ with $k = 0, 1, 2$; the three roots are:

$$z_1 = \sqrt[6]{2}\left(cos\frac{7}{12}\pi + i\,sin\frac{7}{12}\pi\right);$$

$$z_2 = \sqrt[6]{2}\left(cos\frac{5}{4}\pi + i\,sin\frac{5}{4}\pi\right) = -\frac{\sqrt[3]{4}}{2}(1+i);$$

$$z_3 = \sqrt[6]{2}\left(cos\frac{23}{12}\pi + i\,sin\frac{23}{12}\pi\right).$$

By the goniometric bisection formulas we can calculate $cos\frac{7}{12}\pi$ and $sin\frac{7}{12}\pi$;

$$cos\frac{7}{12}\pi = -\sqrt{\frac{1 + cos(7\pi/6)}{2}} = -\sqrt{\frac{1 - \sqrt{3}/2}{2}} = -\frac{1}{2}\left(\sqrt{6} - \sqrt{2}\right)$$ and

$$sin\frac{7}{12}\pi = \sqrt{\frac{1 - cos(7\pi/6)}{2}} = \sqrt{\frac{1 + \sqrt{3}/2}{2}} = \frac{1}{2}\left(\sqrt{6} + \sqrt{2}\right)$$, hence

$z_1 = -\frac{\sqrt[3]{4}}{2}\left(\left(\sqrt{3} - 1\right) - \left(\sqrt{3} + 1\right)i\right)$; while for $cos\frac{23}{12}\pi$ and $sin\frac{23}{12}\pi$ we get:

$$cos\frac{23}{12}\pi = \sqrt{\frac{1 + cos(23\pi/6)}{2}} = \sqrt{\frac{1 + cos(11\pi/6)}{2}} = \sqrt{\frac{1 + \sqrt{3}/2}{2}} = \frac{1}{2}\left(\sqrt{6} + \sqrt{2}\right)$$ and

$$sin\frac{23}{12}\pi = -\sqrt{\frac{1 - cos(23\pi/6)}{2}} = -\sqrt{\frac{1 - cos(11\pi/6)}{2}} = -\sqrt{\frac{1 - \sqrt{3}/2}{2}} = -\frac{1}{2}\left(\sqrt{6} - \sqrt{2}\right)$$

and in conclusion $z_3 = \frac{\sqrt[3]{4}}{2}\left(\left(\sqrt{3} + 1\right) - \left(\sqrt{3} - 1\right)i\right)$.

2) Consider the matrix: $\mathbb{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & k & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Knowing that its determinant is 2; find the value of k, and with the given value of k calculate its inverse.

The determinant of $\mathbb{A}$ is $|\,\mathbb{A}\,| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & k & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} k & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = k - 1 + 1 = k$; put $|\,\mathbb{A}\,| = 2$ we get $k = 2$, $\mathbb{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. The inverse of $\mathbb{A}$ is the matrix $\dfrac{1}{|\,\mathbb{A}\,|} \cdot (Adj(\mathbb{A}))^T$ where $Adj(\mathbb{A})$ is the adjoint matrix of $\mathbb{A}$, $Adj(\mathbb{A}) = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \end{bmatrix} =$

$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ and $\mathbb{A}^{-1} = \dfrac{1}{2} \cdot \begin{bmatrix} 1 & 1 & -2 \\ -1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1 & 0 & 1 \end{bmatrix}$.

An alternative procedure to calculate the inverse matrix of $\mathbb{A}$ is by elementary operations on the rows of $\mathbb{A}$, with this procedure:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_3 - R_1}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \overset{\frac{1}{2}R_2}{\longrightarrow}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \overset{R_1 - R_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \overset{R_1 + \frac{1}{2}R_3}{\longrightarrow}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \overset{R_2 - \frac{1}{2}R_3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

. In the last three columns we can read the inverse of $\mathbb{A}$.

3) Given a linear map $F\colon \mathbb{R}^4 \to \mathbb{R}^4$, with $F(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3, x_2 + x_4, x_1 + x_3 - x_4, x_2 + x_4)$. Calculate the matrix associated at F and find the dimention of the immage and the dimention of the kernel of F.

The linear map F is from $\mathbb{R}^4$ to $\mathbb{R}^4$ and $\mathbb{M}_F$, the matrix associated at F must be a 4×4 matrix. The first coordinate of the immage of F is $x_1 + x_2 + x_3$, hence the first row of $\mathbb{M}_F$ is $(\,1 \quad 1 \quad 1 \quad 0\,)$, the second coordinate of the immage of F is $x_2 + x_4$, hence the second row of $\mathbb{M}_F$ is $(\,0 \quad 1 \quad 0 \quad 1\,)$ and so on follow $\mathbb{M}_F = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. For the dimention of the immage and the dimention of the kernel remember that the dimention of the immage is the rank of matrix $\mathbb{M}_F$, while the dimention of the kernel is the difference between the dimention of dominion of F and the dimention of the immage, the rank of $\mathbb{M}_F$ can be calculated by elementary operations

on rows of $\mathbb{M}_F$: $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \overset{R_3 - R_1}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \overset{\substack{R_3 + R_2 \\ R_4 - R_2}}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. As we can note, two rows of reduced matrix have all zeros while the 2×2 submatrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has determinant different from zero; $rank(\mathbb{M}_F) = dim(Imm(F)) = 2$,
$dim(Ker(F)) = dim(\mathbb{R}^4) - dim(Imm(F)) = 4 - 2 = 2$.

4) Consider the matrix: $\mathbb{A} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$. Calculate its eigenvalues and for any eigenvalue find a base for its associated eigenspace.

The characteristic polinomial of $\mathbb{A}$ is $p_{\mathbb{A}}(\lambda) = \begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} =$ $(1-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 4) = (1-\lambda)(\lambda-2)(\lambda+2)$; put $p_{\mathbb{A}}(\lambda) = 0$ we have the three eigenvalues of matrix $\mathbb{A}$: $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -2$. Take a generic vector $v \in \mathbb{R}^3$, it is an eigenvector associated to the eigenvalue λ_1 if $(\mathbb{A} - \mathbb{I})v = \mathbb{O}$ or in system form is $\begin{cases} -v_1 + 2v_3 = 0 \\ 0 = 0 \\ 2v_1 - v_3 = 0 \end{cases} \Rightarrow v_1 = v_3 = 0 \Rightarrow v = \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} = v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. A base for the eigenspace associated to the eigenvalue λ_1 is $B_{\lambda_1} = \{(0 \quad 1 \quad 0)\}$. For λ_2 we get $(\mathbb{A} - 2\mathbb{I})v = \mathbb{O} \Rightarrow \begin{cases} -2v_1 + 2v_3 = 0 \\ -v_2 = 0 \\ 2v_1 - 2v_3 = 0 \end{cases} \Rightarrow \begin{cases} v_3 = v_1 \\ v_2 = 0 \end{cases} \Rightarrow v = \begin{pmatrix} v_1 \\ 0 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $B_{\lambda_2} = \{(1 \quad 0 \quad 1)\}$. Finally for λ_3, $(\mathbb{A} + 2\mathbb{I})v = \mathbb{O} \Rightarrow \begin{cases} 2v_1 + 2v_3 = 0 \\ 3v_2 = 0 \\ 2v_1 + 2v_3 = 0 \end{cases} \Rightarrow$ $\begin{cases} v_3 = -v_1 \\ v_2 = 0 \end{cases} \Rightarrow v = \begin{pmatrix} v_1 \\ 0 \\ -v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $B_{\lambda_2} = \{(1 \quad 0 \quad -1)\}$.