# UNIVERSITA' DEGLI STUDI DI SIENA
# Scuola di Economia e Management
# A.A. 2023/24
# Quantitative Methods for Economic Applications - Mathematics for Economic Applications
# Task 8/1/2024

I M 1) Given the complex number $z = \left(1 - i^3\right)^2$, calculate its cubic roots.
$z = \left(1 - i^3\right)^2 = (1+i)^2 = 1 + 2i + i^2 = 2i$. Putting the the complex number $z$ in goniometric form, we get $z = 2\left(cos\frac{\pi}{2} + i\, sin\frac{\pi}{2}\right)$; for the cubic roots we apply the classical formula: $z_k = \sqrt[3]{2}\left(cos\left(\frac{\pi}{6} + \frac{2}{3}k\pi\right) + i\, sin\left(\frac{\pi}{6} + \frac{2}{3}k\pi\right)\right)$ $k = 0, 1, 2$. The three roots are:

$k = 0 \to z_0 = \sqrt[3]{2}\left(cos\frac{\pi}{6} + i\, sin\frac{\pi}{6}\right) = \sqrt[3]{2}\left(\frac{1}{2}\sqrt{3} + \frac{1}{2}\,i\right) = \frac{\sqrt[3]{2}}{2}\left(\sqrt{3} + \ i\right)$;

$k = 1 \to z_1 = \sqrt[3]{2}\left(cos\frac{5}{6}\pi + i\, sin\frac{5}{6}\pi\right) = \sqrt[3]{2}\left(-\frac{1}{2}\sqrt{3} + \frac{1}{2}\,i\right) = -\frac{\sqrt[3]{2}}{2}\left(\sqrt{3} - \ i\right)$;

$k = 2 \to z_2 = \sqrt[3]{2}\left(cos\frac{3}{2}\pi + i\, sin\frac{3}{2}\pi\right) = -\sqrt[3]{2}\,i$.

I M 2) Given the matrix $\mathbb{A} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & -9 & -3 \end{bmatrix}$. Calculate its eigenvalues and study if the matrix $\mathbb{A}$ is diagonalizable or not.

At the first step we calculate the characteristic polynomial of matrix $\mathbb{A}$;

$P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda - 3 & -1 & -1 \\ -2 & \lambda - 4 & -2 \\ -1 & 9 & \lambda + 3 \end{vmatrix} =$

$(\lambda - 3)\begin{vmatrix} \lambda - 4 & -2 \\ 9 & \lambda + 3 \end{vmatrix} + + \begin{vmatrix} -2 & -2 \\ -1 & \lambda + 3 \end{vmatrix} - \begin{vmatrix} -2 & \lambda - 4 \\ -1 & 9 \end{vmatrix} =$

$(\lambda - 3)((\lambda - 4)(\lambda + 3) + 18) - 2(\lambda + 3) + -2 + 18 - (\lambda - 4) =$

$(\lambda - 3)(\lambda^2 - \lambda + 6) - 2\lambda - 6 + 16 - \lambda + 4 =$

$\lambda^3 - 4\lambda^2 + 9\lambda - 18 - 3\lambda + 14 = \lambda^3 - 4\lambda^2 + 6\lambda - 4 = (\lambda - 2)(\lambda^2 - 2\lambda + 2) =$

$(\lambda - 2)\left((\lambda - 1)^2 + 1\right)$. Putting $P_{\mathbb{A}}(\lambda) = 0$ we find the three eigenvalues of matrix $\mathbb{A}$; if $\lambda - 2 = 0$, we have the first eigenvalue $\lambda_1 = 2$; if $(\lambda - 1)^2 + 1 = 0$ follow $(\lambda - 1)^2 = -1$ and $\lambda - 1 = \pm i$, thus $\lambda_{2,3} = 1 \pm i$. The three eigenvalues are all differents, thus matrix $\mathbb{A}$ is a diagonalizable one.

I M 3) Given a linear map $F\colon \mathbb{R}^4 \to \mathbb{R}^3$, with $F(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 + x_4, 2x_1 + mx_3 + x_4, -x_1 + 2x_2 - x_3)$; knowing that the dimention of its image and the dimention of its kernel are equal, find the value of parameter $m$ and calculate a basis for the image of linear map $F$.

If image and kernel have the same dimention, both are equal to 2, therefore the rank of the matrix associated at $F$ must be 2. Matrix $\mathbb{A}_F$ associated at $F$ is

$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & m & 1 \\ -1 & 2 & -1 & 0 \end{bmatrix}$; we reduce $\mathbb{A}_F$ by elementary operations on its lines:

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & m & 1 \\ -1 & 2 & -1 & 0 \end{bmatrix} \begin{matrix} R_2 \mapsto R_2 - 2R_1 \\ R_3 \mapsto R_3 + R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -4 & m & -1 \\ 0 & 4 & -1 & 1 \end{bmatrix} R_3 \mapsto R_3 + R_2$$

$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -4 & m & -1 \\ 0 & 0 & m-1 & 0 \end{bmatrix}$, from the reduced matrix we find that the unique value of $m$ such that $\mathbb{A}_F$ has rank two is $m = 1$ and so $F(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 + x_4, 2x_1 + x_3 + x_4, -x_1 + 2x_2 - x_3)$. For the basis of the image of $F$, note that the first element of the image $x_1 + 2x_2 + x_4$ is just equal to the sum of the second element $2x_1 + x_3 + x_4$ and the third $-x_1 + 2x_2 - x_3$, thus any element of the image can be written as $(y_2 + y_3, y_2, y_3) = y_2(1, 1, 0) + y_3(1, 0, 1)$ and a basis for the image is the set $\mathcal{B}_{IF} = \{(1, 1, 0), (1, 0, 1)\}$.

I M 4) Given the three vectors $V_1 = (1, 1, 1)$, $V_2 = (1, 0, 1)$ and $V_3 = (1, 1, 0)$; if vector $W$ has coordinates $(1, 2, 0)$ respect to the canonical basis $\mathcal{B} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, what are the coordinates of $W$respect to the basis $\mathcal{B}' = \{V_1, V_2, V_3\}$?

Respect basis $\mathcal{B}$, vector $W = 1(1, 0, 0) + 2(0, 1, 0) + 0(0, 0, 1) = (1, 2, 0)$, while respect basis $\mathcal{B}'$, vector $W = \alpha(1, 1, 1) + \beta(1, 0, 1) + \gamma(1, 1, 0) = (\alpha + \beta + \gamma, \alpha + \gamma, \alpha + \beta)$; putting $(1, 2, 0) = (\alpha + \beta + \gamma, \alpha + \gamma, \alpha + \beta)$ we easily find $\alpha = \gamma = 1$ and $\beta = -1$.

II M 1) Given the equation $f(x, y) = x \cdot cos^2 y + y \cdot sin^2 x = 0$ satisfied at the point $(0, 0)$, verify that with it an implicit function $x = x(y)$ can be defined and then calculate, for this implicit function, its first and second derivatives $x'(0)$ and $x''(0)$.

$f(0, 0) = 0$, $f'_x = cos^2 y + 2\, y \cdot sin\, x \cdot cos\, x$ and $f'_y = -2\, x \cdot cos\, y \cdot sin\, y + sin^2 x$, with $f'_x(0, 0) = 1$ and $f'_y(0, 0) = 0$. Becouse $f'_x(0, 0) \neq 0$, the equation $f(x, y) = x \cdot cos^2 y + y \cdot sin^2 x = 0$ define a function $x = x(y)$ with $x'(0) = -\dfrac{f'_y(0, 0)}{f'_x(0, 0)} = 0$. For the second order derivative we have $x''(0) = -\dfrac{f''_{y,y}(0, 0)}{f'_x(0, 0)}$ becouse $x'(0) = 0$; $f''_{y,y} = 2\, x\big(sin^2 y - cos^2 y\big)$ and $f''_{y,y}(0, 0) = 0$ thus $x''(0) = 0$.

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x, y) = x^2 + 2\, y \\ \text{u.c.: } x^2 + y^2 \leq 4 \end{cases}$.

The function $f$ is a polynomial, continuos function, the admissible region is a disk with center $(0, 0)$ and radius 2, a bounded and closed set, therefore $f$ presents absolute maximum and minimum in the admissible region. The Lagrangian function is $\mathcal{L}(x, y, \lambda) = x^2 + 2\, y - \lambda(x^2 + y^2 - 4)$ with $\nabla\mathcal{L} = (2x - 2\lambda x, 2 - 2\lambda y, -(x^2 + y^2 - 4))$.

$I^\circ\ CASE$ ($free\ optimization$):

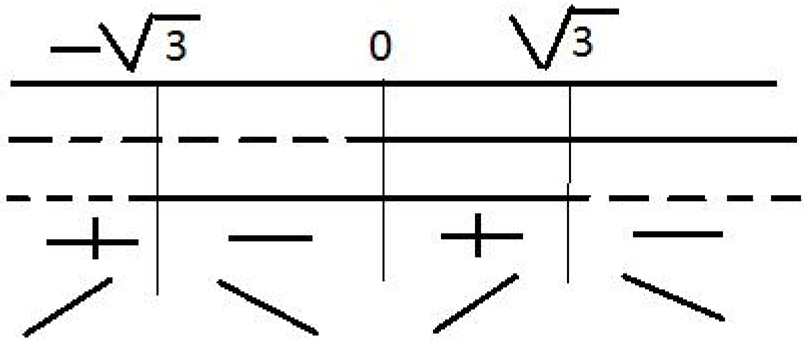
$$\begin{cases} \lambda = 0 \\ 2x = 0 \\ 2 = 0 \\ x^2 + y^2 \le 4 \end{cases}$$ ; system impossible;

$II^\circ\ CASE$ ($constrained\ optimization$):

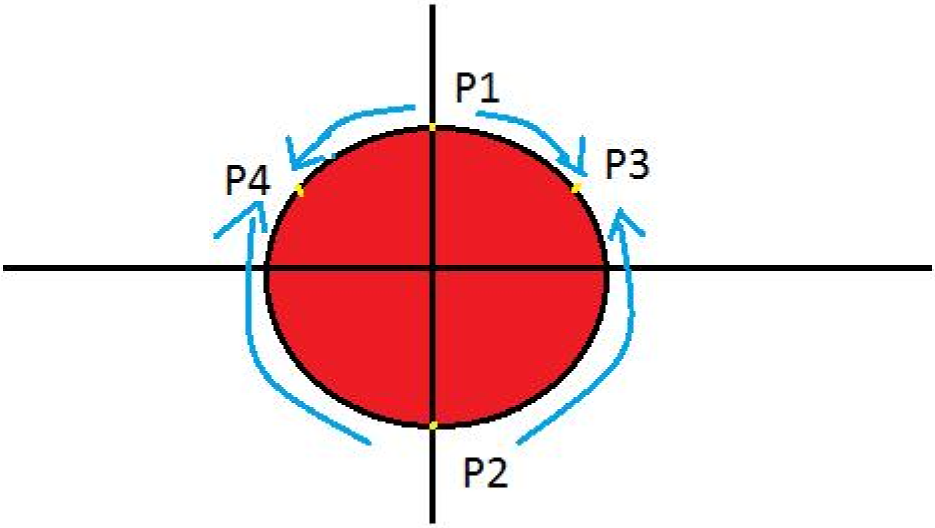
$$\begin{cases} \lambda \ne 0 \\ 2x - 2\lambda x = 0 \\ 2 - 2\lambda y = 0 \\ x^2 + y^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda \ne 0 \\ x(1-\lambda) = 0 \\ \lambda y = 1 \\ x^2 + y^2 = 4 \end{cases}$$ ; if $x = 0$, $y = \pm 2$ and $\lambda = \pm 1/2$, otherwise $\lambda = y = 1$ and $x = \pm\sqrt{3}$. Four critical points $P_{1,2} = (0, \pm 2)$, the first a candidate for maximum ($\lambda > 0$), the second for minimum ($\lambda < 0$), and $P_{3,4} = \left(\pm\sqrt{3}, 1\right)$, both candidate for maximum ($\lambda > 0$). $f(P_{1,2}) = \pm 4$, $f(P_{3,4}) = 5$, $f$ presents absolute maximum equal $5$ on points $\left(\pm\sqrt{3}, 1\right)$ and absolute minimum equal $-4$ on point $(0, -2)$. To analize the nature of point $(0,2)$ we study the function $f$ along the upper border of the admissible region ($y = \sqrt{4-x^2}$).

$$f\left(x, \sqrt{4-x^2}\right) = x^2 + 2\sqrt{4-x^2} = g(x),\ g'(x) = 2x + 2 \cdot \frac{-2x}{2\sqrt{4-x^2}} = \frac{2x\left(\sqrt{4-x^2} - 1\right)}{\sqrt{4-x^2}}$$ ; $g'(x) \ge 0$ if and only if $x\left(\sqrt{4-x^2} - 1\right) \ge 0$:

1. $x \ge 0$;
2. $\sqrt{4-x^2} - 1 \ge 0 \Rightarrow \sqrt{4-x^2} \ge 1 \Rightarrow 4 - x^2 \ge 1 \Rightarrow x^2 \le 3 \Rightarrow -\sqrt{3} \le x \le \sqrt{3}$.




As on the graphic above, $P_{3,4}$ are points of maximum, while $P_1$ is a false maximum. In the graphic in the next page, the admissible region, in red, and the behaviour of $f$ along the border rappresented by the turquoise arrows.




II M 3) Calculate the gradient of the function $f(x,\,y,\,z) = (x+y)^z - log\big(z - y\cdot x^2\big)$ .

$$f'_x = z(x+y)^{z-1} + \frac{2xy}{z - y\cdot x^2}\,; \qquad f'_y = z(x+y)^{z-1} + \frac{x^2}{z - y\cdot x^2}\,;$$

$$f'_z = (x+y)^z \cdot log(x+y) - \frac{1}{z - y\cdot x^2}\,.$$

II M 4) Given the function $f(x,y) = xe^{x+y}$ and the unit vector $v = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ ;

find a point $(x_0, y_0)$ such that $\mathcal{D}_v f(x_0, y_0) = 0$ . After having determined the point $(x_0, y_0)$ , calculate the second order directional derivative $\mathcal{D}^{(2)}_{v,v} f(x_0, y_0)$.

$\nabla f(x,y) = ((1+x)e^{x+y}, xe^{x+y})$, $\mathcal{D}_v f = \nabla f(x,y)\cdot v =$

$((1+x)e^{x+y}, xe^{x+y}) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(1+2x)e^{x+y}$ , thus $\mathcal{D}_v f(x_0, y_0) = 0$ if and only if $1 + 2x = 0$ then $x_0 = -\frac{1}{2}$ . Point $(x_0, y_0)$ that satisfy the condition $\mathcal{D}_v f(x_0, y_0) = 0$ are all the point $\left(-\frac{1}{2}, y_0\right)$ .

$\mathcal{H}f(x,y) = \begin{bmatrix} (2+x)e^{x+y} & (1+x)e^{x+y} \\ (1+x)e^{x+y} & xe^{x+y} \end{bmatrix}$ and

$\mathcal{H}f\left(-\frac{1}{2}, y_0\right) = \begin{bmatrix} \frac{3}{2}e^{y_0-1/2} & \frac{1}{2}e^{y_0-1/2} \\ \frac{1}{2}e^{y_0-1/2} & -\frac{1}{2}e^{y_0-1/2} \end{bmatrix}$ with

$\mathcal{D}^{(2)}_{v,v} f\left(-\frac{1}{2}, y_0\right) = v^T \cdot \mathcal{H}f\left(-\frac{1}{2}, y_0\right)\cdot v =$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \begin{bmatrix} \frac{3}{2}e^{y_0-1/2} & \frac{1}{2}e^{y_0-1/2} \\ \frac{1}{2}e^{y_0-1/2} & -\frac{1}{2}e^{y_0-1/2} \end{bmatrix} \cdot \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} =$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \begin{pmatrix} \sqrt{2}\,e^{y_0-1/2} \\ 0 \end{pmatrix} = e^{y_0-1/2}$.