UNIVERSITA' DEGLI STUDI DI SIENA
Scuola di Economia e Management
A.A. 2023/24
Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 5/2/2024

I M 1) Calculate the complex roots of equation $z^2 + 4i = 0$.
From $z^2 + 4i = 0$ we get $z^2 = -4i$ and finally $z = \sqrt{-4i} = 2\sqrt{-i}$. Putting the immaginary number $-i$ in goniometric form, we get $-i = cos\frac{3\pi}{2} + i\,sin\frac{3\pi}{2}$; for the square roots we apply the classical formula:
$z_k = 2\left(cos\left(\frac{3\pi}{4} + k\pi\right) + i\,sin\left(\frac{3\pi}{4} + k\pi\right)\right)$ $k = 0, 1$. The two roots of equation are:
$k = 0 \to z_0 = 2\left(cos\frac{3\pi}{4} + i\,sin\frac{3\pi}{4}\right) = 2\left(-\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\,i\right) = -\sqrt{2} + \sqrt{2}\,i$;
$k = 1 \to z_1 = 2\left(cos\frac{7\pi}{4} + i\,sin\frac{7\pi}{4}\right) = 2\left(\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\,i\right) = \sqrt{2} - \sqrt{2}\,i = -z_0$.

I M 2) Given the matrix $\mathbb{A} = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & -2 \end{bmatrix}$. Calculate its eigenvalues and study if the matrix $\mathbb{A}$ is diagonalizable or not.
At the first step we calculate the characteristic polynomial of matrix $\mathbb{A}$;
$P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda-2 & -2 & 0 & 0 \\ -2 & \lambda-2 & 0 & 0 \\ 0 & 0 & \lambda+2 & 2 \\ 0 & 0 & 2 & \lambda+2 \end{vmatrix} =$
$\begin{vmatrix} \lambda-2 & -2 \\ -2 & \lambda-2 \end{vmatrix} \cdot \begin{vmatrix} \lambda+2 & 2 \\ 2 & \lambda+2 \end{vmatrix} = \left((\lambda-2)^2 - 4\right) \cdot \left((\lambda+2)^2 - 4\right) =$
$(\lambda^2 - 4\lambda) \cdot (\lambda^2 + 4\lambda) = \lambda^2(\lambda - 4)(\lambda + 4)$. Putting $P_{\mathbb{A}}(\lambda) = 0$ we find three eigenvalues of matrix $\mathbb{A}$. $\lambda_1 = 0$, with algebraic multiplicity 2, $\lambda_2 = 4$, with algebraic multiplicity 1 and $\lambda_3 = -4$, with algebraic multiplicity 1. To verify if the matrix is diagonalizable, we find the geometric multiplicity of eigenvalue λ_1, for our goal we calculate the rank of matrix $\mathbb{A}$, it's easy note that from matrix $\mathbb{A}$ we can define a principal minor of order 2, $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ with determinant different from 0 and matrix $\mathbb{A}$ has the second raw equal to the first and the fourth equal to the third, thus $Rank(\mathbb{A}) = 2$ and the geometric multiplicity of eigenvalue λ_1 is 2. The matrix is diagonalizable.

I M 3) Given the linear system $\begin{cases} x_1+x_2+x_3+x_4=0 \\ mx_1+x_2+x_3+x_4=0 \\ mx_1+mx_2+x_3+x_4=0 \\ mx_1+mx_2+mx_3+x_4=0 \end{cases}$, where m is a real parameter. We indicate with S_m the set of its solutions, study, varing of m, the dimention of the set S_m, and when the dimention is bigger, find a basis for S_m.

The matrix associated to the system is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ m & 1 & 1 & 1 \\ m & m & 1 & 1 \\ m & m & m & 1 \end{bmatrix}$. We reduce the matrix by elementary operations on its lines: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ m & 1 & 1 & 1 \\ m & m & 1 & 1 \\ m & m & m & 1 \end{bmatrix} \begin{matrix} R_2 \mapsto R_2 - mR_1 \\ R_3 \mapsto R_3 - mR_1 \\ R_4 \mapsto R_4 - mR_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1-m & 1-m & 1-m \\ 0 & 0 & 1-m & 1-m \\ 0 & 0 & 0 & 1-m \end{bmatrix}$, from the reduced matrix we find that: $dim(S_m) = \begin{cases} 3 & \text{if } m=1 \\ 0 & \text{if } m \neq 1 \end{cases}$. Dimention of S_m is bigger if $m=1$, and in this case the system is reduced to the unique equazion $x_1+x_2+x_3+x_4=0$ or $x_4 = -x_1-x_2-x_3$, and a generic element of S_m is $(x_1,x_2,x_3,-x_1-x_2-x_3) = x_1(1,0,0,-1)+x_2(0,1,0,-1)+x_3(0,0,1,-1)$, a basis for S_m is the set of vectors $\mathcal{B} = \{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1)\}$.

I M 4) Check if there are values of x and y for which the matrix $\mathbb{A} = \begin{bmatrix} -\frac{13}{3} & x \\ y & \frac{19}{3} \end{bmatrix}$ is similar to the matrix $\mathbb{B} = \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix}$, if the matrix of the similarity transformation is $\mathbb{P} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Matrix $\mathbb{A}$ is similar to the matrix $\mathbb{B}$ with matrix of trasformation $\mathbb{P}$ if $\mathbb{A} = \mathbb{P}^{-1} \cdot \mathbb{B} \cdot \mathbb{P}$.

$|\mathbb{P}| = -3$, $\mathbb{P}^{-1} = \dfrac{1}{|\mathbb{P}|} \cdot \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$ and

$\mathbb{P}^{-1} \cdot \mathbb{B} \cdot \mathbb{P} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} =$

$\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 9 & 6 \\ -2 & -7 \end{bmatrix} = \begin{bmatrix} -\frac{13}{3} & -\frac{20}{3} \\ \frac{20}{3} & \frac{19}{3} \end{bmatrix}$.

Put $\begin{bmatrix} -\frac{13}{3} & x \\ y & \frac{19}{3} \end{bmatrix} = \begin{bmatrix} -\frac{13}{3} & -\frac{20}{3} \\ \frac{20}{3} & \frac{19}{3} \end{bmatrix}$ we find $x = -\dfrac{20}{3}$ and $y = \dfrac{20}{3}$.

II M 1) Given the equation $f(x,y) = x \cdot sin^2 x + y \cdot cos^2 y = 0$ satisfied at the point $(0,0)$, verify that with it an implicit function $y = y(x)$ can be defined and then calculate, for this implicit function, its first and second derivatives $y'(0)$ and $y''(0)$.

$f(0,0) = 0$, $f'_x = 1 \cdot sin^2 x + x \cdot 2 \cdot sin\, x \cdot cos\, x$ and $f'_y = 1 \cdot cos^2 y - y \cdot 2 \cdot cos\, y \cdot sin\, y$, with $f'_x(0,0) = 0$ and $f'_y(0,0) = 1$. Because $f'_y(0,0) \neq 0$, the equation $f(x,y) = x \cdot sin^2 x + y \cdot cos^2 y = 0$ define a function $y = y(x)$ with $y'(0) = -\dfrac{f'_x(0,0)}{f'_y(0,0)} = 0$. For the second order derivative we have

$y''(0) = -\dfrac{f''_{x,x}(0,0)}{f'_y(0,0)}$ because $y'(0) = 0$;

$f''_{x,x} = 2 \cdot sin\, x \cdot cos\, x\, + 2 \cdot sin\, x \cdot cos\, x + x \cdot 2 \cdot cos^2\, x - x \cdot 2 \cdot sin^2\, x$ and $f''_{x,x}(0,0) = 0$ thus $y''(0) = 0$.

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x^2 + y^2 \\ \text{u.c.: } x^2 + 4y^2 + 8y \le 12 \end{cases}$.

The function f is a polynomial, continuos function, the admissible region is the interior region of an ellipse, a bounded and closed set; constraint is qualified on ellipse therefore f presents absolute maximum and minimum in the admissible region. The Lagrangian function is

$\mathcal{L}(x,y,\lambda) = x^2 + y^2 - \lambda(x^2 + 4y^2 + 8y - 12)$ with

$\nabla\mathcal{L} = (2x - 2\lambda x, 2y - 8\lambda y - 8\lambda, -(x^2 + 4y^2 + 8y - 12))$.

$I^\circ\ CASE\ (free\ optimization)$:

$$\begin{cases} \lambda = 0 \\ 2x = 0 \\ 2y = 0 \\ x^2 + 4y^2 + 8y - 12 \le 0 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ x = 0 \\ y = 0 \\ -12 \le 0 \end{cases}$$; point $(0,0)$ is admissible,

$\mathcal{H}f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathcal{H}_1 = 2 > 0$, $\mathcal{H}_2 = 4 > 0$. $(0,0)$ is a minimum point.

$II^\circ\ CASE\ (constrained\ optimization)$:

$$\begin{cases} \lambda \ne 0 \\ 2x - 2\lambda x = 0 \\ 2y - 8\lambda y - 8\lambda = 0 \\ x^2 + 4y^2 + 8y - 12 = 0 \end{cases} \Rightarrow \begin{cases} \lambda \ne 0 \\ 2x(1-\lambda) = 0 \\ 2(y - 4\lambda y - 4\lambda) = 0 \\ x^2 + 4y^2 + 8y = 12 \end{cases}$$; if $x = 0$ and $y = -3$,

$\lambda = 3/8$, if $x = 0$ and $y = 1$, $\lambda = 1/8$, otherwise $\lambda = 1$, $y = -4/3$ and $x = \pm 2\sqrt{35}/3$. Four critical points $P_1 = (0, -3)$, $P_2 = (0,1)$ and $P_{3,4} = \left(\pm 2\sqrt{35}/3, -4/3\right)$, all candidates for maximum ($\lambda > 0$). $f(P_1) = 9$, $f(P_2) = 1$, $f(P_{3,4}) = 52/3 > 9$, f presents absolute maximum equal $52/3$ on points $\left(\pm 2\sqrt{35}/3, -4/3\right)$ and absolute minimum equal 0 on point $(0,0)$. To analize the nature of point P_1 and P_2 we study the function f along the upper and the lower border of the admissible region. Rewrite the ellipse's equation as $x^2 + 4(y+1)^2 = 16$, we get $(y+1)^2 = 4 - \frac{1}{4}x^2$ and the upper and the lower border of the admissible region are respectively $y = -1 + \sqrt{4 - \frac{1}{4}x^2}$ and $y = -1 - \sqrt{4 - \frac{1}{4}x^2}$.

In the upper border consider the function $f\left(x, -1 + \sqrt{4 - \frac{1}{4}x^2}\right) =$

$$x^2 + \left(-1 + \sqrt{4 - \frac{1}{4}x^2}\right)^2 = 5 + \frac{3}{4}x^2 - \sqrt{16 - x^2} = g(x),$$

$$g'(x) = \frac{3}{2}x - \frac{-2x}{2\sqrt{16 - x^2}} = \frac{x\left(3\sqrt{16 - x^2} + 2\right)}{2\sqrt{16 - x^2}}$$; $g'(x) \ge 0$ if and only if $x \ge 0$.

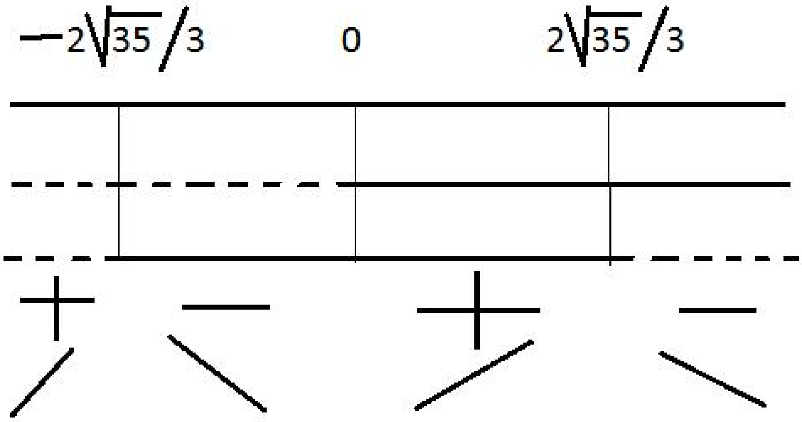
Along the upper border, function f is decreasing for $x \ge 0$ and increasing for $x \ge 0$, P_2 is a false maximum (minimum point along the border).

In the lower border consider the function $f\left(x, -1 - \sqrt{4 - \frac{1}{4}x^2}\right) =$

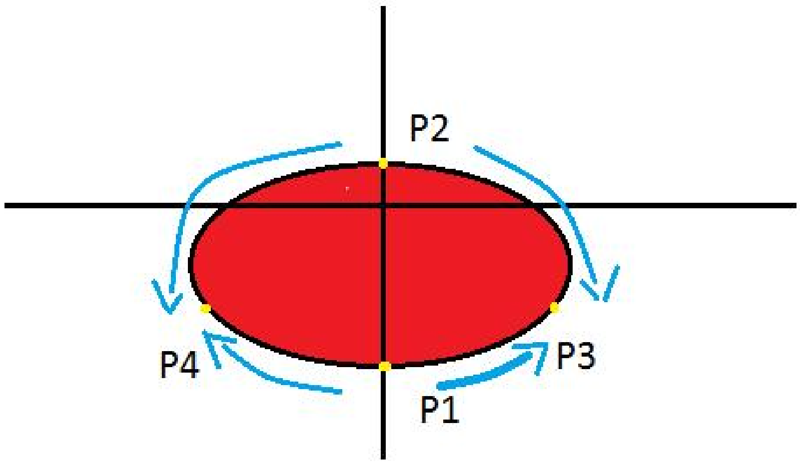
$$x^2 + \left(-1 - \sqrt{4 - \frac{1}{4}x^2} \right)^2 = 5 + \frac{3}{4}x^2 + \sqrt{16 - x^2} = h(x),$$

$h'(x) = \dfrac{3}{2}x + \dfrac{-2x}{2\sqrt{16 - x^2}} = \dfrac{x\left(3\sqrt{16 - x^2} - 2\right)}{2\sqrt{16 - x^2}}$; $h'(x) \geq 0$ if and only if $x\left(3\sqrt{16 - x^2} - 2\right) \geq 0$.

1. $x \geq 0$;
2. $3\sqrt{16 - x^2} - 2 \geq 0 \Rightarrow \sqrt{16 - x^2} \geq \dfrac{2}{3} \Rightarrow 16 - x^2 \geq \dfrac{4}{9} \Rightarrow x^2 \leq \dfrac{140}{9} \Rightarrow$ $-\dfrac{2}{3}\sqrt{35} \leq x \leq \dfrac{2}{3}\sqrt{35}$.



As on the graphic above, $P_{3,4}$ are maximum points, while P_1 is a false maximum (minimum point along the border). In the graphic below, the admissible region, in red, and the behaviour of f along the border rappresented by the turquoise arrows.



II M 3) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x + y \\ \text{u.c.: } x^2 + y^2 = \frac{18}{4} \end{cases}$.

The Lagrangian function of the problem is

$\mathcal{L}(x, y, \lambda) = x + y - \lambda\left(x^2 + y^2 - \dfrac{18}{4}\right)$ with

$$\nabla\mathcal{L} = \left(1 - 2\lambda x, 1 - 2\lambda y, -\left(x^2 + y^2 - \frac{18}{4}\right)\right).$$

FOC:

$$\begin{cases} 1-2\lambda x=0 \\ 1-2\lambda y=0 \\ x^2+y^2=\frac{18}{4} \end{cases} \Rightarrow \begin{cases} x=1/(2\lambda) \\ y=1/(2\lambda) \\ 1/(4\lambda^2)+1/(4\lambda^2)=\frac{18}{4} \end{cases} \Rightarrow \begin{cases} x=1/(2\lambda) \\ y=1/(2\lambda) \\ 2/\lambda^2=18 \end{cases} \Rightarrow \begin{cases} x=\pm\frac{3}{2} \\ y=\pm\frac{3}{2} \\ \lambda=\pm\frac{1}{3} \end{cases};$$

two constraint critical points $P_{1,2}=\left(\pm\dfrac{3}{2},\pm\dfrac{3}{2},\right)$.

SOC:

$$\overline{\mathcal{H}}=\begin{bmatrix} 0 & -2x & -2y \\ -2x & -2\lambda & 0 \\ -2y & 0 & -2\lambda \end{bmatrix}, \text{ with } \left|\overline{\mathcal{H}}\right|=\begin{vmatrix} 0 & -2x & -2y \\ -2x & -2\lambda & 0 \\ -2y & 0 & -2\lambda \end{vmatrix}=$$

$$-2\lambda\cdot\begin{vmatrix} 0 & -2x \\ -2x & -2\lambda \end{vmatrix}-2y\cdot\begin{vmatrix} -2x & -2\lambda \\ -2y & 0 \end{vmatrix}=8\lambda x^2+8\lambda y^2=8\lambda(x^2+y^2).$$

$\left|\overline{\mathcal{H}}(P_1)\right|=12>0$, P_1 point of maximum with $f(P_1)=3$, $\left|\overline{\mathcal{H}}(P_2)\right|=-12<0$, P_2 point of minimum and $f(P_2)=-3$.

II M 4) Given the function $f(x,y)=(x+y)e^{x-y}$ and the unit vector $v=\left(\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)$; calculate on point $(0,0)$ the directional derivatives $\mathcal{D}_v f(0,0)$ and $\mathcal{D}^{(2)}_{v,v}f(0,0)$.

$\nabla f(x,y)=((1+x+y)e^{x-y},(1-x-y)e^{x-y})$, $\nabla f(0,0)=(1,1)$,

$$\mathcal{D}_v f(0,0)=\nabla f(0,0)\cdot v=(1,1)\cdot\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)=\sqrt{2}.$$

$$\mathcal{H}f(x,y)=\begin{bmatrix} (2+x+y)e^{x-y} & -(x+y)e^{x-y} \\ -(x+y)e^{x-y} & -(2-x-y)e^{x-y} \end{bmatrix} \text{ and } \mathcal{H}f(0,0)=\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

with $\mathcal{D}^{(2)}_{v,v}f(0,0)=v^T\cdot\mathcal{H}f(0,0)\cdot v=\left(\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)\cdot\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\cdot\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}=$

$$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\cdot\begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}=0.$$