UNIVERSITA' DEGLI STUDI DI SIENA
Facoltà di Economia "R. Goodwin"
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Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 18/3/2024

I M 1) Find all the complex numbers z such that their immaginary part are equal 2 and the module of complex number $z+i$ is equal 5. For every complex number z found, calculate its argument.
If the immaginary part of complex number is equal 2, $z=a+2i$ and $z+i=a+3i$ with the module $\rho=\sqrt{a^2+9}$. Put $\sqrt{a^2+9}=5$ follow $a^2+9=25$ and $a^2=16$ with $a=\pm 4$; the request complex numbers are $z_1=4+2i$ and $z_2=-4+2i$. For its arguments remember that if $a\neq 0$, the argument of z can be calculated by $\theta=arctg\left(\frac{b}{a}\right)$ and for z_1 and z_2 we get $\theta_{1,2}=arctg\left(\pm\frac{2}{4}\right)=\pm arctg\left(\frac{1}{2}\right)$; $\theta_{1,2}\approx\pm 0.46$ radiant.

I M 2) Given the matrix $\mathbb{A}=\begin{bmatrix}2&2&-1\\-1&2&1\\3&k&-2\end{bmatrix}$ and knowing that 1 is an eigenvalue of the matrix; study if the matrix $\mathbb{A}$ is diagonalizable or not.
At the first step we calculate the characteristic polynomial of matrix $\mathbb{A}$;
$P_{\mathbb{A}}(\lambda)=|\lambda\mathbb{I}-\mathbb{A}|=\begin{vmatrix}\lambda-2&-2&1\\1&\lambda-2&-1\\-3&-k&\lambda+2\end{vmatrix}=(\lambda-2)\cdot\begin{vmatrix}\lambda-2&-1\\-k&\lambda+2\end{vmatrix}+$
$-2\cdot\begin{vmatrix}1&-1\\-3&\lambda+2\end{vmatrix}-\begin{vmatrix}1&\lambda-2\\-3&-k\end{vmatrix}=(\lambda-2)\cdot((\lambda-2)(\lambda+2)-k)+$
$-2\cdot(\lambda+2-3)-(-k+3(\lambda-2))=(\lambda-2)\cdot(\lambda^2-4-k)-2\cdot(\lambda-1)+$
$-(3\lambda-6-k)=\lambda^3-2\lambda^2-(9+k)\lambda+16+3k$. If 1 is an eigenvalue of the matrix, 1 is a root of the characteristic polynomial thus $P_{\mathbb{A}}(1)=0$ and $P_{\mathbb{A}}(1)=6+2k$; put $6+2k=0$ easily we find $k=-3$. Matrix $\mathbb{A}=\begin{bmatrix}2&2&-1\\-1&2&1\\3&-3&-2\end{bmatrix}$ and $P_{\mathbb{A}}(\lambda)=\lambda^3-2\lambda^2-6\lambda+7=(\lambda-1)\cdot(\lambda^2-\lambda-7)$.
Now we calculate the remaining two eigenvalues putting $\lambda^2-\lambda-7=0$, the equation has solutions $\frac{1\pm\sqrt{29}}{2}$ and the three eigenvalues of $\mathbb{A}$ are $\lambda_1=1$ and $\lambda_{2,3}=\frac{1\pm\sqrt{29}}{2}$; matrix is diagonalizable becouse its three eigenvalues are one to one distinct.

I M 3) Given the linear system $\begin{cases}mx_1+mx_2+mx_3=0\\mx_1+mx_2+x_3=0\\mx_1+x_2+x_3=0\end{cases}$, where m is a real parameter. We indicate with S_m the set of its solutions, study, varing of m, the dimention of the set S_m, and when the dimention is bigger, find a basis for S_m.

The matrix associated to the system is $\begin{bmatrix} m & m & m \\ m & m & 1 \\ m & 1 & 1 \end{bmatrix}$. We reduce the matrix by elementary operations on its lines:

$$\begin{bmatrix} m & m & m \\ m & m & 1 \\ m & 1 & 1 \end{bmatrix} \underset{R_3 \mapsto R_3 - R_1}{R_2 \mapsto R_2 - R_1} \begin{bmatrix} m & m & m \\ 0 & 0 & 1-m \\ 0 & 1-m & 1-m \end{bmatrix} R_2 \circlearrowleft R_3 \begin{bmatrix} m & m & m \\ 0 & 1-m & 1-m \\ 0 & 0 & 1-m \end{bmatrix}.$$

The determinant of the reduced matrix is $\begin{vmatrix} m & m & m \\ 0 & 1-m & 1-m \\ 0 & 0 & 1-m \end{vmatrix} = m(1-m)^2$ and it is different from zero if and only if $m \neq 0$ and $m \neq 1$, rank of matrix $\begin{bmatrix} m & m & m \\ m & m & 1 \\ m & 1 & 1 \end{bmatrix}$ is three if and only if $m \neq 0$ and $m \neq 1$ and this imply that in this case the dimention of S_m is zero. If $m = 1$ the matrix associated to the system is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, with rank equal at 1 and dimention of S_m 2; finally if $m = 0$ the matrix associated to the system is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, with rank equal at 2 and dimention of S_m 1. In conclusion $dim(S_m) = \begin{cases} 2 & \text{if } m = 1 \\ 1 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$. Dimention of S_m is bigger if $m = 1$, and in this case the system is reduced to the unique equazion $x_1 + x_2 + x_3 = 0$ or $x_3 = -x_1 - x_2$, and a generic element of S_1 is $(x_1, x_2, -x_1 - x_2) = x_1(1, 0, -1) + x_2(0, 1, -1)$, a basis for S_1 is the set of vectors $\mathcal{B}_{S_1} = \{(1, 0, -1), (0, 1, -1)\}$. To be thorough we consider also the case $m = 0$, in this situation the system is reduced to $\begin{cases} x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = 0 \\ x_2 = 0 \end{cases}$, and a generic element of S_0 is $(x_1, 0, 0) = x_1(1, 0, 0)$, a basis for S_0 is the set $\mathcal{B}_{S_0} = \{(1, 0, 0)\}$.

I M 4) Given a linear map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$, we know that:

1. $F(1,1,1) = (0,0,0)$;
2. $F(0,0,1) = (0,0,1)$;
3. $F(1,0,0) = (1,0,0)$.

Find the dimention of its image and the dimention of its kernel; and for both, image and kernel, set a basis.

For the linear map F we know that the vector $(1,1,1)$ belongs in the kernel of F, thus the dimention of the kernel is at least one: $dim(KerF) \geq 1$. The two linear indipendent vectors $(0,0,1)$ and $(1,0,0)$ belong in the image of F, thus the dimention of the image is at least two: $dim(ImaF) \geq 2$. By the dimention Theorem is known that for the linear map F, $dim(KerF) + dim(ImaF) = dim(\mathbb{R}^3) = 3$, an by the two previous inequalities easily we conclude that $dim(KerF) = 1$ and $dim(ImaF) = 2$. The two basis for the spaces kernel and image can be easily found as $\mathcal{B}_{KerF} = \{(1,1,1)\}$ and $\mathcal{B}_{ImaF} = \{(0,0,1), (1,0,0)\}$.

II M 1) The equation $f(x,y) = 0$ satisfied on point $P(x_P, y_P)$, defined an implicit function $y = y(x)$. We know that the gradient of f on point P is $\nabla f(P) = (1, -1)$ and the Hessian matrix of f on point P is $\mathcal{H}f(P) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. For this implicit function calculate the first and second derivatives $y'(x_P)$ and $y''(x_P)$.

By the Dini's Theorem if $f'_y(P) \neq 0,\ y'(x_P) = -\dfrac{f'_x(P)}{f'_y(P)} = -\dfrac{1}{-1} = 1$; and

$$y''(x_P) = -\frac{f''_{x,x}(P) + 2 \cdot f''_{x,y}(P) \cdot y'(x_P) + f''_{y,y}(P) \cdot (y'(x_P))^2}{\left(f'_y(P)\right)^2} =$$
$$-\frac{f''_{x,x}(P) + 2 \cdot f''_{x,y}(P) + f''_{y,y}(P)}{\left(f'_y(P)\right)^2} =$$
$$-\frac{1 + 2 \cdot (-1) + 1}{(-1)^2} = 0\,.$$

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x^3 + y^3 \\ \text{u.c.: } x^2 + y^2 \leq 8 \end{cases}$.

The function f is a polynomial, continuos function, the admissible region is the interior region of a circunference, a bounded and closed set; constraint is qualified on circunference, therefore f presents absolute maximum and minimum in the admissible region. The Lagrangian function is
$\mathcal{L}(x,y,\lambda) = x^3 + y^3 - \lambda(x^2 + y^2 - 8)$ with
$\nabla\mathcal{L} = (3x^2 - 2\lambda x, 3y^2 - 2\lambda y, -(x^2 + y^2 - 8))$.

$I^\circ\ CASE\ (free\ optimization)$:

$\begin{cases} \lambda = 0 \\ 3x^2 = 0 \\ 3y^2 = 0 \\ x^2 + y^2 - 8 \leq 0 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ x = 0 \\ y = 0 \\ -8 \leq 0 \end{cases}$; point $(0,0)$ is admissible, $\mathcal{H}f = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$ and

$\mathcal{H}f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathcal{H}_2(0,0) = 0$. We haven't any information about the nature of point $(0,0)$.

$II^\circ\ CASE\ (constrained\ optimization)$:

$\begin{cases} \lambda \neq 0 \\ 3x^2 - 2\lambda x = 0 \\ 3y^2 - 2\lambda y = 0 \\ x^2 + y^2 - 8 = 0 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x(3x - 2\lambda) = 0 \\ y(3y - 2\lambda) = 0 \\ x^2 + y^2 = 8 \end{cases}$; we must evaluate four possibilities:

a: if $x = 0$ and $y = 0$, $0^2 + 0^2 \neq 8$; point $(0,0)$ isn't admissible;

b: if $x = 0$ and $y = \dfrac{2}{3}\lambda$, we get $\dfrac{4}{9}\lambda^2 = 8 \Rightarrow \lambda^2 = 18 \Rightarrow \lambda = \pm 3\sqrt{2}$; point $P_1\left(0, 2\sqrt{2}\right)$ is a candidate to maximum, while point $P_2\left(0, -2\sqrt{2}\right)$ is a candidate to minimum;

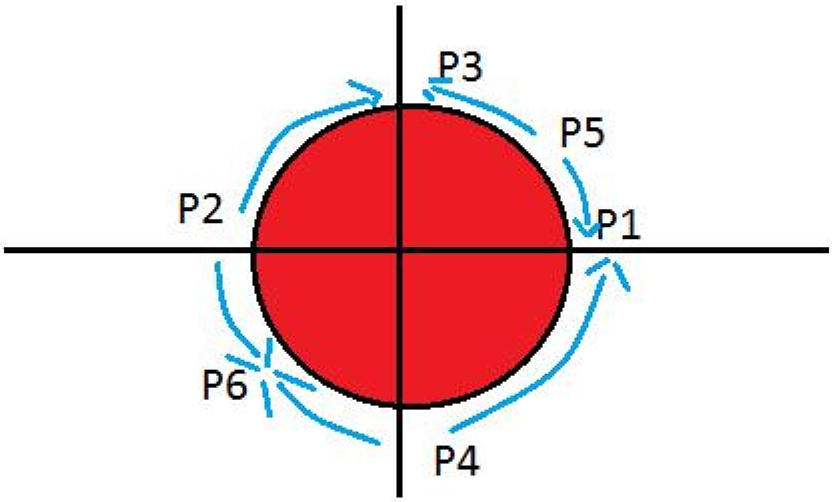
c: if $y = 0$ and $x = \dfrac{2}{3}\lambda$, we get again $\dfrac{4}{9}\lambda^2 = 8 \Rightarrow \lambda = \pm 3\sqrt{2}$; point $P_3\left(2\sqrt{2}, 0\right)$ is a candidate to maximum, while point $P_4\left(-2\sqrt{2}, 0\right)$ is a candidate to minimum;

d: if $x = \dfrac{2}{3}\lambda$ and $y = \dfrac{2}{3}\lambda$, we get $\dfrac{8}{9}\lambda^2 = 8 \Rightarrow \lambda^2 = 9 \Rightarrow \lambda = \pm 3$; point $P_5(2,2)$ is a candidate to maximum, while point $P_6(-2,-2)$ is a candidate to minimum.

$f(P_{1,3}) = 16\sqrt{2}$, $f(P_5) = 16 < 16\sqrt{2}$, f presents absolute maximum equal $16\sqrt{2}$ on points $\left(0, 2\sqrt{2}\right)$ and $\left(2\sqrt{2}, 0\right)$; $f(P_{2,4}) = -16\sqrt{2}$, $f(P_6) = -16 > -16\sqrt{2}$, f presents absoluteminimum equal $-16\sqrt{2}$ on points $\left(0, -2\sqrt{2}\right)$ and $\left(-2\sqrt{2}, 0\right)$.

To analize the nature of point P_5 and P_6 we study the function f along the upper and the lower border of the admissible region. Rewrite the circunference's equation as

$y^2 = 8 - x^2$, the upper and the lower borders of the admissible region are respectively $y = +\sqrt{8-x^2}$ and $y = -\sqrt{8-x^2}$.
In the upper border consider the function $f\left(x, +\sqrt{8-x^2}\right) =$
$x^3 + \left(\sqrt{8-x^2}\right)^3 = g(x), g'(x) = 3x^2 + 3\left(\sqrt{8-x^2}\right)^2 \cdot \dfrac{-2x}{2\sqrt{8-x^2}} =$
$3x\left(x - \sqrt{8-x^2}\right)$; $g'(x) > 0$ if and only if $-2\sqrt{2} < x < 0$ or $2 < x < 2\sqrt{2}$. Along the upper border, function f is increasing for $-2\sqrt{2} < x < 0$ and $2 < x < 2\sqrt{2}$, decreasing for $0 < x < 2$, P_5 is a false maximum (minimum point along the border).
By the exchangeability on variables in function f ($f(x,y) = f(y,x)$), similar results can be achaived in the lower border.
In the graphic below, the admissible region, in red, and the behaviour of f along the border rappresented by the turquoise arrows.



II M 3) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x^2 + y^2 \\ \text{u.c.: } x - y = 4 \end{cases}$.
The Lagrangian function of the problem is
$\mathcal{L}(x,y,\lambda) = x^2 + y^2 - \lambda(x - y - 4)$ with
$\nabla\mathcal{L} = (2x - \lambda, 2y + \lambda, -(x - y - 4))$.
FOC:
$\begin{cases} 2x - \lambda = 0 \\ 2y + \lambda = 0 \\ x - y = 4 \end{cases} \Rightarrow \begin{cases} x = \lambda/2 \\ y = -\lambda/2 \\ \lambda/2 + \lambda/2 = 4 \end{cases} \Rightarrow \begin{cases} x = \lambda/2 \\ y = -\lambda/2 \\ \lambda = 4 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = -2 \\ \lambda = 4 \end{cases}$; one constraint critical points $P = (2, -2)$.
SOC:
$\overline{\mathcal{H}} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, with $|\overline{\mathcal{H}}| = \begin{vmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} =$
$-2 - 2 = -4$. $|\overline{\mathcal{H}}(P)| < 0$, P point of minimum with $f(P) = 8$.

II M 4) Given the function $f(x,y) = e^{x+y} - e^{x-y}$ and the unit vector $v = \left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$;
calculate on point $(0,0)$ the directional derivatives $\mathcal{D}_v f(0,0)$ and $\mathcal{D}^{(2)}_{v,v} f(0,0)$.

$$\nabla f(x,y) = (e^{x+y} - e^{x-y}, e^{x+y} + e^{x-y}),\ \nabla f(0,0) = (0,2),$$

$$\mathcal{D}_v f(0,0) = \nabla f(0,0) \cdot v = (0,2) \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \sqrt{3}.$$

$$\mathcal{H}f(x,y) = \begin{bmatrix} e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \\ e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \end{bmatrix} \text{ and } \mathcal{H}f(0,0) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \text{ with}$$

$$\mathcal{D}^{(2)}_{v,v} f(0,0) = v^T \cdot \mathcal{H}f(0,0) \cdot v = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \cdot \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} =$$

$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \cdot \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \sqrt{3}\,.$$