UNIVERSITA' DEGLI STUDI DI SIENA
Scuola di Economia e Management
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Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 3/6/2024

I M 1) Given the complex number $z = \frac{(1-i)^2}{(1+i)^2}$, calculate its square roots.

$z = \frac{(1-i)^2}{(1+i)^2} = \frac{1-2i+i^2}{1+2i+i^2} = \frac{1-2i-1}{1+2i-1} = \frac{-2i}{2i} = -1 = cos\pi + i\,sin\pi$. For the square roots we apply the classical formula:

$\sqrt{z} = \sqrt{cos\pi + i\,sin\pi} = cos\left(\frac{\pi}{2} + k\pi\right) + i\,sin\left(\frac{\pi}{2} + k\pi\right)$ $k = 0, 1$. The two roots are:

$k = 0 \rightarrow z_0 = cos\left(\frac{\pi}{2}\right) + i\,sin\left(\frac{\pi}{2}\right) = i$;

$k = 1 \rightarrow z_1 = cos\left(\frac{3\pi}{2}\right) + i\,sin\left(\frac{3\pi}{2}\right) = -i$.

I M 2) Given the matrix $\mathbb{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$, calculate its eigenvalues and study if the matrix $\mathbb{A}$ is diagonalizable or not.

At the first step we calculate the characteristic polynomial of matrix $\mathbb{A}$;

$P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda-1 & 0 & -2 \\ 0 & \lambda-1 & 0 \\ -4 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda-1 & -2 \\ -4 & \lambda-1 \end{vmatrix} =$ $(\lambda-1)\left((\lambda-1)^2 - 8\right)$. Putting $P_{\mathbb{A}}(\lambda) = 0$ we find the three eigenvalues of matrix $\mathbb{A}$; if $\lambda - 1 = 0$, we have the first eigenvalue $\lambda_1 = 1$; if $(\lambda-1)^2 - 8 = 0$ it follow $(\lambda-1)^2 = 8$ and $\lambda - 1 = \pm 2\sqrt{2}$, thus $\lambda_{2,3} = 1 \pm 2\sqrt{2}$. The three eigenvalues are one to one differents, thus matrix $\mathbb{A}$ is a diagonalizable one.

I M 3) Given the linear application $F: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, we know that:

1. $F(1,0,0) = (1,0)$; 2. $F(1,1,0) = (1,1)$; 3. $F(1,1,1) = (1,1)$.

Find the matrix associated with the linear application, calculate the dimentions of both, kernel and immage of F, and find a basis for the kernel and a basis for the image.

Define with $\mathbb{A}_F = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ the matrix associated to the linear application, by conditions 1., 2. and 3. we know $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} =$ $\begin{bmatrix} a & a+b & a+b+c \\ d & d+e & d+e+f \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, thus $a = 1$, $b = 0$, $c = 0$, $d = 0$, $e = 1$ and $f = 0$; $\mathbb{A}_F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Matrix $\mathbb{A}_F$ has rank equal 2 and the dimention of the image is 2 while the dimention of the kernel is 1; for the basis of the image we can note that the codomain of F is the set $\mathbb{R}^2$ and the dimention of the image is 2 thus $Ima(F) = \mathbb{R}^2$

and a base for it is the set $\mathcal{B}_{Ima(F)} = \{(1,0),(0,1)\}$. For a basis of the kernel we take a generic element of the domain (x,y,z), its image is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, thus (x,y,z) belongs on the kernel if and only if $x = y = 0$ and we conclude that a generic element of the kernel is $(0,0,z) = z(0,0,1)$. A basis for the kernel is the set $\mathcal{B}_{Ker(F)} = \{(0,0,1)\}$.

I M 4) Vector V has coordinates $(1,1,1)$ respect the basis $\mathcal{B} = \{(0,0,1),(0,1,1),(1,1,1)\}$, and coordinates $(-1,-1,-1)$ respect the basis $\mathcal{B}' = \{(1,0,0),(1,1,0),(x,y,z)\}$. Determine the vector (x,y,z).

If vector V has coordinates $(1,1,1)$ respect the basis $\mathcal{B}$ and coordinates $(-1,-1,-1)$ respect the basis $\mathcal{B}'$, $V = 1\cdot(0,0,1) + 1\cdot(0,1,1) + 1\cdot(1,1,1) = (1,2,3)$ and at the same time $V = -1\cdot(1,0,0) - 1\cdot(1,1,0) - 1\cdot(x,y,z) = (-2-x,-1-y,-z)$. Put $(1,2,3) = (-2-x,-1-y,-z)$ it easily follows $x = y = z = -3$.

II M 1) Given the system of equations $\begin{cases} e^{x+y} - e^{y+z} = 0 \\ x - z + xyz = 0 \end{cases}$ satisfied at the point $P(1,0,1)$; verify that with it an implicit function $z \mapsto (x(z),y(z))$ can be defined and then calculate, for this implicit function, the derivatives $x'(1)$ and $y'(1)$.

Consider the function $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ with $F(x,y,z) = (e^{x+y} - e^{y+z}, x - z + xyz)$ and the jacobian matrix $\mathbb{J} = \begin{bmatrix} e^{x+y} & e^{x+y} - e^{y+z} & -e^{y+z} \\ 1+yz & xz & -1+xy \end{bmatrix}$. Matrix $\mathbb{J}$ at point $P(1,0,1)$ is $\begin{bmatrix} e & 0 & -e \\ 1 & 1 & -1 \end{bmatrix}$ and the minor of its restriction respect variables x and y is $\left|\mathbb{J}(P)\big|_{x,y}\right| = \begin{vmatrix} e & 0 \\ 1 & 1 \end{vmatrix} = e \neq 0$, the proposed system of equations define an implicit function $z \mapsto (x(z),y(z))$ on a neighborhood of the point P. For the derivatives we have $x'(1) = -\dfrac{\begin{vmatrix} -e & 0 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} e & 0 \\ 1 & 1 \end{vmatrix}} = 1$ and $y'(1) = -\dfrac{\begin{vmatrix} e & -e \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} e & 0 \\ 1 & 1 \end{vmatrix}} = 0\,.$

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x\cdot y \\ \text{u.c.: } x^2 + y^2 \leq 4 \end{cases}$.

The function f is a polynomial, continuos function, the admissible region is a disk with center $(0,0)$ and radius 2, a bounded and closed set, therefore f presents absolute maximum and minimum in the admissible region. The Lagrangian function is $\mathcal{L}(x,y,\lambda) = x\cdot y - \lambda(x^2+y^2-4)$ with $\nabla\mathcal{L} = (y - 2\lambda x, x - 2\lambda y, -(x^2+y^2-4))$.

$I^\circ\ CASE\ (free\ optimization)$:

$\begin{cases} \lambda = 0 \\ y = 0 \\ x = 0 \\ x^2+y^2 \leq 4 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ y = 0 \\ x = 0 \\ 0 \leq 4 \end{cases}$; $\mathcal{H}f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with determinant $|\mathcal{H}f| = -1 < 0$, $(0,0)$ is a saddle point.

$II^\circ\ CASE\ (constrained\ optimization)$:

$\begin{cases} \lambda \neq 0 \\ y - 2\lambda x = 0 \\ x - 2\lambda y = 0 \\ x^2 + y^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ y = 2\lambda x \\ x - 4\lambda^2 x = 0 \\ x^2 + y^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ y = 2\lambda x \\ x(1 - 4\lambda^2) = 0 \\ x^2 + y^2 = 4 \end{cases}$; if $x = 0$, $y = 0$ and $x^2 + y^2 \neq 4$; otherwise if $1 - 4\lambda^2 = 0$, $\lambda = \pm 1/2$, $y = \pm x$ and by the condition $x^2 + y^2 = 4$ we get $x = \pm\sqrt{2}$; four constrained critical points $P_1\left(\sqrt{2}, \sqrt{2}\right)$, $P_2\left(-\sqrt{2}, -\sqrt{2}\right)$, $P_3\left(\sqrt{2}, -\sqrt{2}\right)$, $P_4\left(-\sqrt{2}, \sqrt{2}\right)$. Two of these $\left(\pm\sqrt{2}, \pm\sqrt{2}\right)$ have $\lambda > 0$, candidate for maximum, the others $\left(\pm\sqrt{2}, \mp\sqrt{2}\right)$ have $\lambda < 0$, candidate for mimimum, also, by the simmetry of function $f(x, y) = f(-x, -y)$ and $f(-x, y) = f(x, -y) = -f(x, y)$, we get the maximum $f\left(\pm\sqrt{2}, \pm\sqrt{2}\right) = 2$ and the minimum -2 on points $\left(\pm\sqrt{2}, \mp\sqrt{2}\right)$.

II M 3) Given the function $f(x, y) = x^3 + y^3 - 3x - 27y$, find the point of minimum and the point of maximum for function f.

$\nabla f = (3x^2 - 3, 3y^2 - 27)$.

FOC: $\begin{cases} 3x^2 - 3 = 0 \\ 3y^2 - 27 = 0 \end{cases} \Rightarrow \begin{cases} x^2 = 1 \\ y^2 = 9 \end{cases} \Rightarrow \begin{cases} x = \pm 1 \\ y = \pm 3 \end{cases}$; four critical points $P_1(1, 3)$, $P_2(-1, -3)$, $P_3(1, -3)$, $P_4(-1, 3)$.

$\mathcal{H}f = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$; $|\mathcal{H}f| = 36xy$.

SOC: $|\mathcal{H}f(1, 3)| = 108 > 0$, $f''_{xx}(1, 3) = 6 > 0$. $(1, 3)$ point of minimum.

$|\mathcal{H}f(-1, -3)| = 108 > 0$, $f''_{xx}(-1, -3) = -6 < 0$. $(-1, -3)$ point of maximum.

$|\mathcal{H}f(\pm 1, \mp 3)| = -108 < 0$. $(\pm 1, \mp 3)$ saddle points.

II M 4) Function $f(x, y) = x^2 - y^2$ has directional derivatives $\mathcal{D}_v f(x_P, y_P) = 2\sqrt{2}$ and $\mathcal{D}_w f(x_P, y_P) = 0$, where v is the unit vector $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and w is the unit vector $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Find the point (x_P, y_P) and calculate the directional derivative $\mathcal{D}^{(2)}_{v,w} f(x_P, y_P)$.

Function f is differentiable for any point (x, y) with gradient $\nabla f(x, y) = (2x, -2y)$,

$$\mathcal{D}_v f(x_P, y_P) = (2x_P, -2y_P) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \sqrt{2}x_P - \sqrt{2}y_P = (x_P - y_P)\sqrt{2}$$

and

$$\mathcal{D}_w f(x_P, y_P) = (2x_P, -2y_P) \cdot \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\sqrt{2}x_P - \sqrt{2}y_P = -(x_P + y_P)\sqrt{2}.$$

Putting $(x_P - y_P)\sqrt{2} = 2\sqrt{2}$ and $-(x_P + y_P)\sqrt{2} = 0$ it easily follows $x_P = 1$ and $y_P = -1$.

Remember that $\mathcal{D}^{(2)}_{v,w} f(x_P, y_P) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \mathcal{H}f(x_P, y_P) \cdot \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$ and $\mathcal{H}f(x_P, y_P) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$; we get

$$\mathcal{D}^{(2)}_{v,w} f(x_P, y_P) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \cdot \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} =$$

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \left(-\sqrt{2}, -\sqrt{2}\right) = -2.$$