## **UNIVERSITA' DEGLI STUDI DI SIENA Scuola di Economia e Management A.A. 2023/24 Quantitative Methods for Economic Applications - Mathematics for Economic Applications Task 4/7/2024** Economic Applica<br>
k 4/7/2024<br>  $\frac{1-i}{1+i}$ , write the complex

**Mathematics for Economic Applications<br>
Task 4/7/2024**<br>
I M 1) Given the complex number  $z = \frac{1-i}{1+i}$ , write the complex number in goniometric form and calculate its cubic roots. **ECONOMIC Applies**<br> **k** 4/7/2024<br>  $\frac{1-i}{1+i}$ , write the complex<br>
roots.<br>  $i^2 = 2i$ goniometric form and calculate its cubic roots. [M 1) Given the complex number  $z = \frac{1-i}{1+i}$ , write the complex number in<br>goniometric form and calculate its cubic roots.<br> $z = \frac{1-i}{1+i} = \frac{1-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-2i+i^2}{1-i^2} = \frac{-2i}{2} = -i$  (remember that  $i^2 = -1$ ). ) Given the complex number  $z = \frac{1-i}{1+i}$ , write the complex num<br>metric form and calculate its cubic roots.<br> $\frac{1-i}{1+i} = \frac{1-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-2i+i^2}{1-i^2} = \frac{-2i}{2} = -i$  (remembe ) Given the complex number  $z = \frac{1-i}{1+i}$ , write the complex num<br>metric form and calculate its cubic roots.<br> $\frac{1-i}{1+i} = \frac{1-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-2i+i^2}{1-i^2} = \frac{-2i}{2} = -i$  (rememb<br>niometric form:  $z = -i = \cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi$ 2  $-$ 2  $-$ (remember that  $i^2 = -1$ ). I M 1) Given the complex number  $z = \frac{1+i}{1+i}$ , write the complex number in<br>goniometric form and calculate its cubic roots.<br> $z = \frac{1-i}{1+i} = \frac{1-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-2i+i^2}{1-i^2} = \frac{-2i}{2} = -i$  (remember that  $i^2 = -1$ ).<br>In gonio  $=\frac{1-i}{1+i}$ , write the complex num<br>ic roots.<br> $\frac{i^2}{i^2} = \frac{-2i}{2} = -i$  (remember<br> $\frac{3}{2}\pi + i \sin \frac{3}{2}\pi$ . For the cubic root  $1+i$ <br>  $\frac{+i^2}{i^2} = \frac{-2i}{2} = -i$  (remember<br>  $\frac{3}{2}\pi + i \sin \frac{3}{2}\pi$ . For the cubic rooting it sin  $\frac{3}{2}\pi =$ I M 1) Given the complex number  $z = \frac{1}{1+i}$ , write the complex number<br>goniometric form and calculate its cubic roots.<br> $z = \frac{1-i}{1+i} = \frac{1-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-2i+i^2}{1-i^2} = \frac{-2i}{2} = -i$  (remember to<br>In goniometric form:  $z =$  $rac{1-i}{1-i} = \frac{1-2i+i^2}{1-i^2} = \frac{-2i}{2} = -i$  (rememb<br>  $z = -i = \cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi$ . For the cubic r<br>  $z = \sqrt[3]{\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi} =$ <br>  $\left(\frac{3}{2}\pi + 2k\pi\right) = \sqrt{\pi} - 2$ . ts cubic roots.<br>  $\frac{-2i + i^2}{1 - i^2} = \frac{-2i}{2} = -i$  (remed  $\frac{3}{2}\pi + i \sin \frac{3}{2}\pi$ . For the cubic  $\frac{3}{2}\pi + i \sin \frac{3}{2}\pi =$  $\frac{-2i + i}{1 - i^2} = \frac{-2i}{2} = -i \text{ (rema)}$ <br>  $\frac{3}{2}\pi + i \sin \frac{3}{2}\pi$ <br>  $\frac{3}{2}\pi + i \sin \frac{3}{2}\pi =$ <br>  $\frac{3}{2}i\pi + 2i\pi =$ <br>  $\frac{3}{2}i\pi + 2i\pi =$  $\frac{2i + i^2}{1 - i^2} = \frac{-2i}{2} = -i$  (remer<br>  $\frac{3}{2}\pi + i \sin \frac{3}{2}\pi$ . For the cubi-<br>  $\pi + i \sin \frac{3}{2}\pi =$ <br>  $2k\pi$   $\left(\pi - 2, \frac{1}{2}\right)$ If a goniometric form:  $z = -i = \cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi$ . For the cubic roots we apply the<br>
classical formula:  $\sqrt[3]{z} = \sqrt[3]{\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi} =$ <br>  $\cos \left(\frac{\frac{3}{2}\pi + 2k\pi}{3}\right) + i \sin \left(\frac{\frac{3}{2}\pi + 2k\pi}{3}\right) = \cos \left(\frac{\pi}{2} + \frac{2}{3}$ oniometric form:  $z = -i = \cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi$ . For the cubic roots we apply the<br>sical formula:  $\sqrt[3]{z} = \sqrt[3]{\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi} =$ <br> $\left(\frac{\frac{3}{2}\pi + 2k\pi}{3}\right) + i \sin \left(\frac{\frac{3}{2}\pi + 2k\pi}{3}\right) = \cos \left(\frac{\pi}{2} + \frac{2}{3}k\pi\right) + i \sin \left(\frac$ the cubic roots we apply the<br>  $\pi$   $\left(\frac{\pi}{2} + \frac{2}{3}k\pi\right)$ classical formula:  $\sqrt[3]{z} = \sqrt[3]{\cos{\frac{3}{2}\pi}}$ <br>  $\cos{\left(\frac{\frac{3}{2}\pi + 2k\pi}{3}\right)} + i \sin{\left(\frac{\frac{3}{2}\pi + 2k\pi}{3}\right)}$ <br>  $k = 0, 1, 2$ . The three roots are:<br>  $k = 0 \rightarrow z_0 = \cos{\left(\frac{\pi}{2}\right)} + i \sin{\left(\frac{\pi}{2}\right)}$ classical formula:  $\sqrt[3]{z} = \sqrt[3]{\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}} =$ <br>  $\cos \left( \frac{\frac{3}{2}\pi + 2k\pi}{3} \right) + i \sin \left( \frac{\frac{3}{2}\pi + 2k\pi}{3} \right) = \cos \left( \frac{\pi}{2} + \frac{2}{3}k\pi \right)$ <br>  $k = 0, 1, 2$ . The three roots are:<br>  $k = 0 \rightarrow z_0 = \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \$  $k = 0 \rightarrow z_0 = cos\left(\frac{\pi}{2}\right) + i sin\left(\frac{\pi}{2}\right) = i;$  $\cos\left(\frac{2^{n+2n}}{3}\right) + i \sin\left(\frac{2^{n+2n}}{3}\right) = \cos\left(\frac{n}{2} + \frac{1}{3}k\pi\right) + i \sin\left(\frac{n}{2} + \frac{1}{3}k\pi\right)$ <br>  $k = 0, 1, 2$ . The three roots are:<br>  $k = 0 \rightarrow z_0 = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) = i$ ;<br>  $k = 1 \rightarrow z_1 = \cos\left(\frac{7}{6}\pi\right) + i \sin\left(\frac{7}{6}\pi\right) = -$ 3  $\int$   $e^{2\pi i/3}$   $\left( \frac{3}{2} \right)^{2}$   $\cos^2(2 + 3^{n})$   $\cos^2(2 + 3^{n})$ <br>
2. The three roots are:<br>  $z_0 = \cos(\frac{\pi}{2}) + i \sin(\frac{\pi}{2}) = i$ ;<br>  $z_1 = \cos(\frac{7}{6}\pi) + i \sin(\frac{7}{6}\pi) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$ ;<br>  $z_2 = \cos(\frac{11}{2}\pi) + i \sin(\frac{11}{2}\pi) = \frac{\sqrt{3}}{2}$  $k = 0 \rightarrow z_0 = cos(\frac{\pi}{2}) + i sin(\frac{\pi}{2}) = i;$ <br>  $k = 1 \rightarrow z_1 = cos(\frac{7}{6}\pi) + i sin(\frac{7}{6}\pi) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i;$ <br>  $k = 2 \rightarrow z_2 = cos(\frac{11}{6}\pi) + i sin(\frac{11}{6}\pi) = \frac{\sqrt{3}}{2} - \frac{1}{2}i.$ <br>
[0 0 0 0 0]  $z_0 = cos\left(\frac{\pi}{2}\right) + i sin\left(\frac{\pi}{2}\right) = i ;$ <br>  $z_1 = cos\left(\frac{7}{6}\pi\right) + i sin\left(\frac{7}{6}\pi\right) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i ;$ <br>  $z_2 = cos\left(\frac{11}{6}\pi\right) + i sin\left(\frac{11}{6}\pi\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}i .$ <br>  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$  $k = 1 \rightarrow z_1 = cos\left(\frac{7}{6}\pi\right) + i sin\left(\frac{7}{6}\pi\right) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i;$ <br>  $k = 2 \rightarrow z_2 = cos\left(\frac{11}{6}\pi\right) + i sin\left(\frac{11}{6}\pi\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}i.$ <br>
I M 2) Given the matrix  $\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$ , calcul  $\begin{aligned} \sin\left(\frac{1}{6}\pi\right) &= -\frac{1}{2} - \frac{1}{2}i; \\ \sin\left(\frac{11}{6}\pi\right) &= \frac{\sqrt{3}}{2} - \frac{1}{2}i. \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ calculate its eigenvector.} \end{aligned}$  $sin\left(\frac{11}{6}\pi\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ .<br>
0 0 0 0 0<br>
0 1 3 0<br>
0 3 1 0<br>
0 0 0 0<br>
1  $sin\left(\frac{12\pi}{6}\pi\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}i.$ <br>
0 0 0 0 0<br>
0 1 3 0<br>
0 3 1 0<br>
0 0 0 0 0<br>
ot.  $\begin{aligned} i \sin\left(\frac{1}{6}\pi\right) &= -\frac{\sqrt{9}}{2} - \frac{1}{2}i \, ; \\ i \sin\left(\frac{11}{6}\pi\right) &= \frac{\sqrt{3}}{2} - \frac{1}{2}i \, . \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix}, \text{ calculate its eigen} \end{aligned}$  $i \sin\left(\frac{11}{6}\pi\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ .<br>  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ , calculate its eigen  $i \sin \left(\frac{\pi}{6}\pi\right) = \frac{1}{2} - \frac{1}{2}i$ .<br>  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ , calculate its eigen<br>
not. , calculate its eigenvalues and study if the  $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ , calculate its eigenconductions.<br>
the characteristic polynomial of man<br>  $\begin{bmatrix} 0 & 0 & 0 \\ -1 & -3 & 0 \\ 2 & 1 & 0 \end{bmatrix} = \lambda \begin{bmatrix} \lambda - 1 & -1 \\ -3 & \lambda - 1 \end{bmatrix}$ ix  $\mathbb{A} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ , calculate its eigenvalues and study if the<br>
cable or not.<br>
lculate the characteristic polynomial of matrix A;<br>  $\begin{vmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda - 1 & -3 & 0 \\ 0 & -3 & \lambda - 1 & 0 \$ 

matrix  $A$  is diagonalizable or not.

At the first step we calculate the characteristic polynomial of matrix  $\mathbb{A}$ ;

matrix A is diagonalizable or not.  
\nAt the first step we calculate the characteristic polynomial of matrix A;  
\n
$$
P_{\mathbb{A}}(\lambda) = |\lambda \mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda - 1 & -3 & 0 \\ 0 & -3 & \lambda - 1 & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^2 (\lambda - 1)^2 - 9 = \lambda^2 (\lambda^2 - 2\lambda - 8) = \lambda^2 (\lambda - 4)(\lambda + 2).
$$
\nPutting  $P_{\mathbb{A}}(\lambda) = 0$  we find the four eigenvalues of matrix A; if  $\lambda^2 = 0$ , we have the first two eigenvalues  $\lambda_{1,2} = 0$ ; if  $\lambda - 4 = 0$  it follows  $\lambda_3 = 4$  and finally if  $\lambda + 2 = 0$  it follows  $\lambda_4 = -2$ . The eigenvalue zero discloses algebraic multiplicity equal two, thus

two eigenvalues ; if it follows and finally if it  $\lambda^2 \begin{vmatrix} \lambda - 1 & -3 \\ -3 & \lambda - 1 \end{vmatrix} = \lambda^2 ((\lambda - 1)^2 - 9) = \lambda^2 (\lambda^2 - 2\lambda - 8) = \lambda^2 (\lambda - 4)(\lambda + 2)$ .<br>Putting  $P_{\mathbb{A}}(\lambda) = 0$  we find the four eigenvalues of matrix  $\mathbb{A}$ ; if  $\lambda^2 = 0$ , we have the first two eigenvalues  $\lambda_{1,2} =$ matrix  $\mathbb A$  is a diagonalizable one if and only if the geometric multiplicity of the eigenvalue zero is two. To calculate such geometric multiplicity, we calculate the rank

of the matrix  $0 \cdot I - A = -A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & - \\ 0 & -3 & - \\ 0 & 0 & 0 \end{bmatrix}$  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$ ; easily we can not  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ; easily we can not 0 0 0 0 0<br>
0 -1 -3 0<br>
0 -3 -1 0<br>
0 0 0 0<br>
1 1 2  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ; easily we can not<br>the  $\begin{vmatrix} -1 & -3 \\ 2 & 1 \end{vmatrix} = -8$  is different 1  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & -3 & -1 & 0 \end{bmatrix}$ ; easily we can note  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ; easily we can note  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ; easily we can note<br>  $\begin{aligned} \text{ax, the} \begin{vmatrix} -1 & -3 \\ -3 & -1 \end{vmatrix} = -8 \text{ is different from } \mathbb{R}^2. \end{aligned}$ ; easily we can note that only of the matrix  $0 \cdot \mathbb{I} - \mathbb{A} = -\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ; easily we can note that only<br>one two by two minor of the matrix, the  $\begin{vmatrix} -1 & -3 \\ -3 & -1 \end{vmatrix} = -8$  is different from zer

conclude that the geometric multiplicity of the eigenvalue zero is two (the difference one two by two minor of the matrix, the  $\begin{vmatrix} -1 & -3 \\ -3 & -1 \end{vmatrix} = -8$  is different from zero, we<br>conclude that the geometric multiplicity of the eigenvalue zero is two (the difference<br>between the order of the matrix A, 4, one two by two minor of the matrix, the  $\begin{vmatrix} 1 & 0 \\ -3 & -1 \end{vmatrix} = -8$  is d<br>conclude that the geometric multiplicity of the eigenvalue zero is t<br>between the order of the matrix A, 4, and the rank of the matrix 0<br>is a digona conclude that the geometric multiplicity of the eigenvalue zero is two (the difference<br>between the order of the matrix A, 4, and the rank of the matrix  $0 \cdot \mathbb{I} - A$ , 2). Matrix A<br>is a digonalizable one.<br>I M 3) Given the

is a digonalizable one.<br>I M 3) Given the linear application  $F: \mathbb{R}^3 \to \mathbb{R}^3$ , with

$$
F(x_1, x_2, x_3) = (x_1 + x_2, x_1 + x_2 + x_3, x_2 + x_3).
$$

kernel and immage of  $F$ , and find a basis for the image.

IM 3) Given the linear application  $F: \mathbb{R}^3 \to \mathbb{R}^3$ , with<br>  $F(x_1, x_2, x_3) = (x_1 + x_2, x_1 + x_2 + x_3, x_2 + x_3)$ .<br>
Find the matrix associated with the linear application, calculate the dimentions of both,<br>
kernel and immage Find the matrix associated with the linear application, calculate the dimentions of k<br>
Find the matrix associated with the linear application, calculate the dimentions of k<br>
kernel and immage of F, and find a basis for th  $x_2, x_1 + x_2 + x_3, x_2 + x_3$ .<br>
ated with the linear application, calculate the dimentions of both,<br>
F, and find a basis for the image.<br>  $+ x_2, x_1 + x_2 + x_3, x_2 + x_3$ , the matrix associated to the linear<br>  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 &$  $x_2, x_1 + x_2 + x_3, x_2 + x_3$ .<br>
ated with the linear application, calculate the dimentions of both,<br>
F, and find a basis for the image.<br>  $+ x_2, x_1 + x_2 + x_3, x_2 + x_3$ , the matrix associated to the linear<br>  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 &$ F, and find a basis for the image.<br>  $+x_2, x_1 + x_2 + x_3, x_2 + x_3$ ), the matrix associated to the linear<br>  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ . The determinant of matrix  $\mathbb{A}_F$  is  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ +  $x_2$ ,  $x_1$  +  $x_2$  +  $x_3$ ,  $x_2$  +  $x_3$ ).<br>ciated with the linear application<br>of *F*, and find a basis for the ima<br> $x_1 + x_2$ ,  $x_1 + x_2 + x_3$ ,  $x_2 + x_3$ ).<br> $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ . The determinant c of F, and find a basis for the image.<br>  $x_1 + x_2, x_1 + x_2 + x_3, x_2 + x_3$ , the matrix associated<br>  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ . The determinant of matrix  $\mathbb{A}_F$  is  $\begin{vmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{vmatrix}$ <br>  $0 - 1 = -1 \neq 0$ dimentions of both,<br>
pociated to the linear<br>  $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} =$ If  $F(x_1, x_2, x_3) = (x_1 + x_2, x_1 + x_2 + x_3, x_2 + x_3)$ , the matrix associated to the linear<br>application is  $\mathbb{A}_F = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ . The determinant of matrix  $\mathbb{A}_F$  is  $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 &$ application is  $\mathbb{A}_F = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ . The determinant of matrix  $\mathbb{A}_F$  is  $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 0 - 1 = -1 \neq 0$ . Matrix  $\mathbb{A}_F$  has rank three  $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  = 0 - 1 = -1  $\neq$  0. Matrix  $\mathbb{A}_F$  has rank that the dimention of the Image of *F* is 3, while the dimention dim( $\mathbb{R}^3$ ) -  $dim(Ima(F))$  $dim(\mathbb{R}^3) - dim(Ima(F)) = 3 - 3 = 0$ . Because the codomain of the linear application is  $\mathbb{R}^3$  and dimention of the Image is again 3, easily follows that  $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$  -  $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$  = 0 - 1 =<br>that the dimention of the Imag<br> $dim(\mathbb{R}^3) - dim(Ima(F))$  =<br>application is  $\mathbb{R}^3$  and dimentic<br> $Ima(F) = \mathbb{R}^3$  and a basis for<br> $B_{Ima(F)} = B_{\mathbb{R}^3} = \{(1,0,0), (0,1)\$  $Ima(F) = \mathbb{R}^3$  and a basis for the Image is the set <sup>1</sup> <sup>1</sup> <sup>1</sup> <sup>1</sup> <sup>0</sup> <sup>1</sup> <sup>1</sup> <sup>1</sup> <sup>0</sup> <sup>1</sup> <sup>1</sup> <sup>1</sup> that the dimention of the Image of *F* is 3, while the dimention of the Kernel is  $dim(\mathbb{R}^3) - dim(Ima(F)) = 3 - 3 = 0$ . Because the codomain of the linear application is  $\mathbb{R}^3$  and  $Im(a|X) = 0$  and  $\overline{X}$  and dimention of the Image is again 3, easily follows that<br>  $Im(a(F) = \mathbb{R}^3$  and a basis for the Image is the set<br>  $\mathcal{B}_{Ima(F)} = \mathcal{B}_{\mathbb{R}^3} = \{(1,0,0), (0,1,0), (0,0,1)\}.$ <br>
I M 4) Vector V has coord approached is as and dimension of the lingte is again 5, easily follows that  $Ima(F) = \mathbb{R}^3$  and a basis for the Image is the set  $\mathcal{B}_{Ima(F)} = \mathcal{B}_{\mathbb{R}^3} = \{(1,0,0), (0,1,0), (0,0,1)\}\$ .<br>
IM 4) Vector V has coordinates  $(1,1$  $B_{Ima(F)} = B_{\mathbb{R}^3} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$ <br>
I M 4) Vector *V* has coordinates  $(1, 1, 1)$  respect the basis  $B = \{(0, 0, 1), (0, 1, 1), (1, 1, 1)\}$ ; find<br>
the coordinates of vector *V* respect the basis  $B' = \{(1, 1, 1), (1$ respect the basis  $\mathcal{B}', V = 1 \cdot (0, 0, 1) + 1 \cdot (0, 1, 1) + 1 \cdot (1, 1, 1) =$ as coordinates  $(1, 1, 1)$  respect the basis  $B = \{(0, 0, 1), (0, 1, 1), (1, 0), (1), (0, 1, 0)\}$ . Is the <br>0, 1),  $(0, 1, 0)$  a basis for the vector space  $\mathbb{R}^3$ ?<br>ordinates  $(1, 1, 1)$  respect the basis B and coordinates  $(\alpha, \beta$ FM 4) Vector V has coolumates  $(1, 1, 1)$  respect the basis  $B = \{(0, 0, 1), (0, 1, 1), (1, 1, 1)\}$ , the<br>the coordinates of vector V respect the basis  $B' = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ . Is the set<br> $C = \{(1, 1, 1), (1, 0, 1), (0, 1,$ the coordinates of vector  $V$  respect the basis  $B = \{(1, 1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ . Is the set  $C = \{(1, 1, 1), (1, 0, 1), (0, 1, 0)\}$  a basis for the vector space  $\mathbb{R}^3$ ?<br>If vector  $V$  has coordinates  $(1, 1, 1)$  respect Let  $\alpha$  the second part of the exercise, note that<br>  $\alpha$  respect the basis  $\beta'$ ,  $V = 1 \cdot (0, 0, 1) + 1 \cdot (0, 1, 1) + 1 \cdot (1, 1, 1) =$ <br>  $(1, 2, 3)$  and at the same time  $V = \alpha \cdot (1, 1, 1) + \beta \cdot (1, 1, 0) + \chi \cdot (1, 0, 0)$ <br>  $(\alpha + \beta + \chi,$ If vector  $V$  has coordinates  $(1, 1, 1)$  respect the basis  $B$  and coordinates  $(\alpha, \beta, \chi)$ <br>respect the basis  $B'$ ,  $V = 1 \cdot (0, 0, 1) + 1 \cdot (0, 1, 1) + 1 \cdot (1, 1, 1) =$ <br> $(1, 2, 3)$  and at the same time  $V = \alpha \cdot (1, 1, 1) + \beta \cdot (1,$ (1, 2, 3) and at the same time  $V = \alpha \cdot (1, 1, 1) + \beta \cdot (1, 1, 0) + \chi \cdot (1, 0, 0) =$ <br>
( $\alpha + \beta + \chi, \alpha + \beta, \alpha$ ). Putting  $(1, 2, 3) = (\alpha + \beta + \chi, \alpha + \beta, \alpha)$  it easily follows  $\alpha = 3$  and  $\beta = \chi = -1$ . For the second part of the exercise, n  $(1, 2, 3)$  and at the same time  $V = \alpha \cdot (1, 1, 1) + \beta \cdot (1, 1, 0) + \chi \cdot (1, 0, 0) =$ <br>  $(\alpha + \beta + \chi, \alpha + \beta, \alpha)$ . Putting  $(1, 2, 3) = (\alpha + \beta + \chi, \alpha + \beta, \alpha)$  it easily follows  $\alpha = 3$  and  $\beta = \chi = -1$ . For the second part of the exercise,  $(a - b \tan \beta - \lambda - 1)$ . Then the second part of the exercuse, note that<br>  $(1, 0, 1) + (0, 1, 0) = (1, 1, 1)$ , thus the set C is a set of three linear dependent vectors<br>
belonging to  $\mathbb{R}^3$  and easily we conclude that C isn't a belonging to  $\mathbb{R}^3$  and easily we conclude that C isn't a basis for the vector space  $\mathbb{R}^3$ .<br>
II M 1) Given the equation  $x \cdot e^{x+y-z^2} - z^2 \cdot e^{x+y+z} = 0$  satisfied at the point  $P(1,0,-1)$ ; verify that with it an impli  $P(1,0,-1)$ ; verify that with it an implicit function  $(x, y) \mapsto z(x, y)$  can b<br>and then calculate, for this implicit function, the partial derivatives  $z'_x$  and z<br>Consider the function  $f: \mathbb{R}^3 \to \mathbb{R}$  with  $f(x, y, z) = x \cdot e^{x$ nd then calculate, for this implicit function, the partial derivatives  $z'_x$  and  $z'_y$ .<br>
Consider the function  $f: \mathbb{R}^3 \to \mathbb{R}$  with  $f(x, y, z) = x \cdot e^{x+y-z^2} - z^2 \cdot e^{x+y+z}$ .<br>  $f'(P) = 1 \cdot e^0 - 1 \cdot e^0 = 0$ , the partial derivativ Consider the function  $f: \mathbb{R}^3 \to \mathbb{R}$  with  $f(x, y, z) = x \cdot e^{x+y-z} - z^2 \cdot e^{x+y+z}$ .<br>  $f(P) = 1 \cdot e^0 - 1 \cdot e^0 = 0$ , the partial derivative of f respect the variable z is<br>  $f'_z = x \cdot e^{x+y-z^2} \cdot (-2z) - 2z \cdot e^{x+y+z} - z^2 \cdot e^{x+y+z} =$ <br>  $-z(2x \cdot e$  $f(P) = 1 \cdot e^0 - 1 \cdot e^0 = 0$ , the partial derivative of f respect the variable z is<br>  $f'_z = x \cdot e^{x+y-z^2} \cdot (-2z) - 2z \cdot e^{x+y+z} - z^2 \cdot e^{x+y+z} =$ <br>  $-z \left(2x \cdot e^{x+y-z^2} + (2+z)e^{x+y+z}\right)$ ; on point  $P(1,0,-1)$  the partial derivative has<br>
value  $f'_z$  $f'_z = x \cdot e^{x+y-z^2} \cdot (-2z) - 2z \cdot e^{x+y+z} - z^2 \cdot e^{x+y+z} =$ <br>  $- z \left( 2x \cdot e^{x+y-z^2} + (2+z)e^{x+y+z} \right)$ ; on point  $P(1,0,-1)$  the partial derivative has<br>
value  $f'_z(P) = 1 \cdot (2 \cdot e^0 + 1 \cdot e^0) = 3 \neq 0$ ; the proposed equation defines in a<br>
neighbou  $\frac{1}{w}$ : value  $f'_z(P) = 1 \cdot (2 \cdot e^0 + 1 \cdot e^0) = 3 \neq 0$ ; the proposed equation defines in a<br>neighbourhood of point *P* an implicit function  $(x, y) \mapsto z(x, y)$ . To calculate the<br>partial derivatives  $z'_x$  and  $z'_y$  we must firstly calculate neighbourhood of point *P* an implicit function  $(x, y) \mapsto z(x, y)$ <br>partial derivatives  $z'_x$  and  $z'_y$  we must firstly calculate the two pa<br> $f'_y$ :<br> $f'_x = 1 \cdot e^{x+y-z^2} + x \cdot e^{x+y-z^2} - z^2 \cdot e^{x+y+z} = (1+x) \cdot e^{x+y-z}$ <br> $f'_y = x \cdot e^{x+y-z^2} - z^2 \cdot$ 

with  $f'_x(P) = 2 \cdot e^0 - 1 \cdot e^0 = 1$  and  $f'_y(P) = f(P) = 0$ . The two partial derivatives of  $f'_x(P) = 2 \cdot e^0 - 1 \cdot e^0 = 1$  and  $f'_y(P) = f(P) = 0$ . The two partial<br>on z are  $z'_x(1,0) = -\frac{f'_x(P)}{f'(P)} = -\frac{1}{3}$  and  $z'_y(1,0) = -\frac{f'_y(P)}{f'(P)}$ with  $f'_x(P) = 2 \cdot e^0 - 1 \cdot e^0 = 1$  and  $f'_y(P) = f(P) = 0$ . The two partial derivatives of<br>function z are  $z'_x(1,0) = -\frac{f'_x(P)}{f'_z(P)} = -\frac{1}{3}$  and  $z'_y(1,0) = -\frac{f'_y(P)}{f'_z(P)} = -\frac{0}{3} = 0$ .<br>HM2) Salso the partition  $\int \text{Max/min } f(x,y) = x + y$ = 1 and  $f'_y(P) = f(P) = 0$ . The two partial derivatives of<br> $\frac{f'_x(P)}{f'_z(P)} = -\frac{1}{3}$  and  $z'_y(1,0) = -\frac{f'_y(P)}{f'_z(P)} = -\frac{0}{3} = 0$ . = 1 and  $f'_y(P) = f(P) = 0$ . The two partial derivatives of<br>  $\frac{f'_x(P)}{f'_z(P)} = -\frac{1}{3}$  and  $z'_y(1,0) = -\frac{f'_y(P)}{f'_z(P)} = -\frac{0}{3} = 0$ .<br>
Max/min  $f(x,y) = x + y$  $e^{0} - 1 \cdot e^{0} = 1$  and  $f'_{y}(P) = f(P) = 0$ . The two partial derivatives of<br>  $f'_{x}(1,0) = -\frac{f'_{x}(P)}{f'_{z}(P)} = -\frac{1}{3}$  and  $z'_{y}(1,0) = -\frac{f'_{y}(P)}{f'_{z}(P)} = -\frac{0}{3} = 0$ .<br>
(Max/min  $f(x, y) = x + y$ 1 and  $f'_y(P) = f(P) = 0$ . The two partial derivat<br>  $\frac{f'_x(P)}{f'_z(P)} = -\frac{1}{3}$  and  $z'_y(1,0) = -\frac{f'_y(P)}{f'_z(P)} = -\frac{0}{3}$ <br>
(ax/min  $f(x, y) = x + y$  $e^{0} - 1 \cdot e^{0} = 1$  and  $f'_{y}(P) = f(P) = 0$ . The two partial deri<br>  $(1, 0) = -\frac{f'_{x}(P)}{f'_{z}(P)} = -\frac{1}{3}$  and  $z'_{y}(1, 0) = -\frac{f'_{y}(P)}{f'_{z}(P)} = -$ <br>
problem  $\begin{cases} \text{Max/min } f(x, y) = x + y \\ 0 < 0 \end{cases}$ . and  $f'_y(P) = f(P) = 0$ . The two partial derivatives of<br>  $\frac{(P)}{(P)} = -\frac{1}{3}$  and  $z'_y(1,0) = -\frac{f'_y(P)}{f'_z(P)} = -\frac{0}{3} = 0$ .<br>  $x/\min f(x, y) = x + y$ <br>  $\therefore 4x^2 + y^2 \le 4$ . partial derivatives of<br>  $\frac{(P)}{(P)} = -\frac{0}{3} = 0$ . II M 2) Solve the problem  $\begin{cases} \text{Max/min } f(x, y) = x + y \\ \text{u.c.: } 4x^2 + y^2 \le 4 \end{cases}.$  $f'_y(P) = f(P) = 0$ . The two partial c<br>  $-\frac{1}{3}$  and  $z'_y(1,0) = -\frac{f'_y(P)}{f'_z(P)} =$ <br>  $f(x,y) = x + y$ <br>  $+y^2 \le 4$ <br>
to function, the admissible region is and  $f'_y(P) = f(P) = 0$ . The two<br>  $\left(\frac{P}{P}\right)^2 = -\frac{1}{3}$  and  $z'_y(1,0) = -\frac{f'_y}{f'_z}$ <br>  $\left(\begin{array}{c}\text{min } f(x,y) = x+y\\4x^2 + y^2 \le 4\end{array}\right)$ <br>  $4x^2 + y^2 \le 4$ <br>
antinuos function, the admissible is

The function  $f$  is a polynomial, continuos function, the admissible region is an ellipse function z are  $z'_x(1,0) = -\frac{3x}{f'_z(P)} = -\frac{1}{3}$  and  $z'_y(1,0) = -\frac{3x+3}{f'_z(P)} = -\frac{1}{3} = 0$ .<br>
II M 2) Solve the problem  $\begin{cases} \text{Max/min } f(x,y) = x+y \\ \text{u.c.: } 4x^2 + y^2 \le 4 \end{cases}$ .<br>
The function f is a polynomial, continuos function, th therefore  $f$  presents absolute maximum and minimum in the admissible region. The Lagrangian function is The function f is a polynomial, continuos function, the adr<br>with center (0, 0) and axises of lengths equal to 2 and 4, a b<br>therefore f presents absolute maximum and minimum in th<br>Lagrangian function is<br> $\mathcal{L}(x, y, \lambda) = x + y -$ 

The function *f* is a polynomial, continuos function, the admissible<br>with center (0, 0) and axises of lengths equal to 2 and 4, a bounded<br>therefore *f* presents absolute maximum and minimum in the admis-<br>Lagrangian functi with center (0, 0) and axises of lengths equal to 2 are<br>therefore f presents absolute maximum and minimu<br>Lagrangian function is<br> $\mathcal{L}(x, y, \lambda) = x + y - \lambda(4x^2 + y^2 - 4)$  with<br> $\nabla \mathcal{L} = (1 - 8\lambda x, 1 - 2\lambda y, -(4x^2 + y^2 - 4)).$ <br> $I^{\circ} CASE$  $\bigwedge \lambda = 0$  $\begin{cases} 1 & \text{if } \\ 1 & \text{if } \end{cases}$  $1 = 0$ Lagrangian function is<br>  $\mathcal{L}(x, y, \lambda) = x + y - \lambda(4x^2)$ <br>  $\nabla \mathcal{L} = (1 - 8\lambda x, 1 - 2\lambda y,$ <br>  $I^{\circ} \ CASE$  (free optimizal)<br>  $\begin{cases} \lambda = 0 \\ 1 = 0 \\ 1 = 0 \end{cases}$ . System in<br>  $\begin{cases} 1 = 0 \\ 4x^2 + y^2 \le 4 \end{cases}$ grangian function is<br>  $x, y, \lambda$  =  $x + y - \lambda(4x^2)$ <br>  $\mathcal{L} = (1 - 8\lambda x, 1 - 2\lambda y,$ <br>  $CASE$  (free optimizat<br>  $\lambda = 0$ <br>  $1 = 0$  System im x,  $y, \lambda$  =  $x + y = \lambda$ (4x<br>  $\mathcal{L} = (1 - 8\lambda x, 1 - 2\lambda y,$ <br>  $CASE$  (free optimizar<br>  $\lambda = 0$ <br>  $1 = 0$  System in<br>  $4x^2 + y^2 \le 4$ CASE (free optimization):<br>  $\lambda = 0$ <br>  $1 = 0$  System impossible<br>  $4x^2 + y^2 \le 4$ <br>  $\alpha$  CASE (constrained optimize) . System impossible.  $\begin{cases}\n\lambda = 0 \\
1 = 0 \\
1 = 0\n\end{cases}$ . System impossible.<br>  $\begin{cases}\n\lambda = 0 \\
1 = 0 \\
4x^2 + y^2 \le 4 \\
II^{\circ} \ \ \text{CASE} \ \text{(constrained optimization)} \\
\begin{cases}\n\lambda \neq 0 \\
\lambda \neq 0\n\end{cases}$  $\begin{cases}\n1 = 0 & . \text{ System impossible.} \\
1 = 0 & . \text{System impossible.} \\
4x^2 + y^2 \le 4 & \text{If } 0 & . \text{} \\
1^{\circ} \text{ CASE} \text{ (constrained optimization):} \\
\begin{cases}\n\lambda \neq 0 \\
1 - 8\lambda x = 0 \\
1 - 2\lambda y = 0\n\end{cases} \Rightarrow\n\begin{cases}\n\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
y = \frac{1}{2\lambda}\n\end{cases} \Rightarrow\n\begin{cases}\n\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
y = \frac{1}{2\lambda}\n\end{cases} \Rightarrow \begin{cases$  $\begin{cases} 4x^2 + y^2 \le 4 \\ II^{\circ} \ \text{CASE} \ \text{(constrained optimization)}: \\ \begin{cases} \lambda \neq 0 \\ 1 - 8\lambda x = 0 \\ 1 - 2\lambda y = 0 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ 4x^2 + y^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ 4\left(\frac{1}{8\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 4 \end{cases} \Rightarrow \frac{$  $(4x^2 + y^2 \le 4)$ <br>
II° CASE (constrained optimization):<br>  $\begin{cases} \lambda \neq 0 \\ 1 - 8\lambda x = 0 \\ 1 - 2\lambda y = 0 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ 4(x^2 + y^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ 4(\frac{1}{8\lambda})^2 + (\frac{1}{2\lambda})^2 = 4$  $4x^2 + y^2 = 4$ stem impossible.<br>
ained optimization):<br>  $\begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{8\lambda} \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{8\lambda} \end{cases} \Rightarrow$ ained optimization):<br>  $\begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ 4\left(\frac{1}{8\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 4 \\ \neq 0 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 4 \end{cases} \Rightarrow$ dified optimization).<br>  $\begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ 4\left(\frac{1}{8\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 4 \\ \neq 0 \\ -\frac{1}{2\lambda} \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 4 \\ \frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 4 \end{cases} \Rightarrow$  $\lambda \neq 0$  $\lambda x = 0$ = 0<br>
= 0<br>  $x^2 + y^2 \le 4$ <br>  $CASE (constrained optimization):$ <br>  $\neq 0$ <br>  $- 8\lambda x = 0$ <br>  $- 2\lambda y = 0$   $\Rightarrow$   $\begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \end{cases}$   $\Rightarrow$   $\begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \end{cases}$  $1 = 0$  System impossion<br>  $4x^2 + y^2 \le 4$ <br>  $\circ$  CASE (constrained optimix<br>  $\lambda \neq 0$ <br>  $1 - 8\lambda x = 0$ <br>  $1 - 2\lambda y = 0$ <br>  $\Rightarrow$ <br>  $\begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \end{cases}$  $4x^2 + y^2 \le 4$ <br>
° CASE (constrained optimi.<br>  $\lambda \ne 0$ <br>  $1 - 8\lambda x = 0$ <br>  $1 - 2\lambda y = 0$   $\Rightarrow$   $\begin{cases} \lambda \ne 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ 4(x^2 + y^2 = 4 \end{cases}$  $^{\circ}$  CASE (constrained optimiz<br>  $\lambda \neq 0$ <br>  $1 - 8\lambda x = 0$ <br>  $1 - 2\lambda y = 0$   $\Rightarrow$   $\begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ 4(x^2 + y^2 = 4) \\ \lambda \neq 0 \end{cases}$   $\Rightarrow$   $\begin{cases} \lambda \neq 0 \\ 4(\frac{1}{8\lambda})^2 + (\frac{1}{2}) \\ \lambda \neq 0 \end{cases}$ impossible.<br> *d* optimization):<br>  $\neq 0$ <br>  $=\frac{1}{8\lambda}$   $\Rightarrow \infty$ m impossible.<br>  $\lambda \neq 0$ <br>  $x = \frac{1}{8\lambda}$ <br>  $y = \frac{1}{2\lambda}$  = red optimization):<br>  $\lambda \neq 0$ <br>  $x = \frac{1}{8\lambda}$ <br>  $y = \frac{1}{2\lambda}$ <br>  $4(\frac{1}{8\lambda})^2 + (\frac{1}{2\lambda})^2 = 4$ ned optimization):<br>  $\lambda \neq 0$ <br>  $x = \frac{1}{8\lambda}$ <br>  $y = \frac{1}{2\lambda}$ <br>  $4(\frac{1}{8\lambda})^2 + (\frac{1}{2\lambda})^2 = 4$ <br>  $\begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 0 \end{cases}$  $\neq 0$ <br>=  $\frac{1}{8\lambda}$   $\Rightarrow$  $\lambda \neq 0$ <br>  $x = \frac{1}{8\lambda}$ <br>  $y = \frac{1}{2\lambda}$   $\Rightarrow$  $\lambda \neq 0$ <br>  $x = \frac{1}{8\lambda}$ <br>  $y = \frac{1}{2\lambda}$ <br>  $\frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 4$ CASE (constrained optimization):<br>  $\neq 0$ <br>  $- 8\lambda x = 0$ <br>  $- 2\lambda y = 0$   $\Rightarrow$   $\begin{cases} x = \frac{1}{8\lambda} \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ 4(\frac{1}{8\lambda})^2 + (\frac{1}{2\lambda})^2 = 4 \end{cases}$   $\Rightarrow$   $\begin{cases} \lambda \neq 0 \\ y = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 4 \$  $\frac{1}{2}$  and  $\frac{1}{2}$  $8\lambda$  $\frac{1}{2}$  and  $\frac{1}{2}$  $2\lambda$  $\begin{cases}\n\frac{1}{8\lambda} \\
\frac{1}{8\lambda} \\
\frac{1}{2\lambda}\n\end{cases} \Rightarrow \begin{cases}\n\frac{1}{8\lambda} \\
\frac{1}{8\lambda}\lambda^2 + \left(\frac{1}{2\lambda}\right)^2 = 4 \\
\lambda + 0\n\end{cases}$  $=\frac{1}{8\lambda}$ <br>  $=\frac{1}{2\lambda}$ <br>  $=\frac{1}{2\lambda}$ <br>  $\left(\lambda \neq 0\right)$ <br>  $\left(\lambda \neq 0\right)$  $\frac{1}{\lambda}$   $\frac{1}{\lambda}$   $\frac{1}{2} + \left(\frac{1}{2\lambda}\right)^2 = 4$   $\frac{1}{2}$   $\frac{1}{\sqrt{2}}$  $8\lambda$  $\frac{1}{\sqrt{2}}$  $2\lambda$  $\neq 0$ <br>  $= \frac{1}{8\lambda}$ <br>  $= \frac{1}{2\lambda}$ <br>  $\Rightarrow$ <br>  $\frac{1}{3\lambda^2} + \frac{1}{4\lambda^2} = 4$  $x = \frac{1}{8\lambda}$ <br>  $y = \frac{1}{2\lambda}$ <br>  $\frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 4$ strained optimization):<br>  $\Rightarrow \begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ 4\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 4 \end{cases} \Rightarrow$  $\lambda$  $\lambda$  $\begin{cases}\n\frac{1}{8\lambda} \\
\frac{1}{2\lambda} \\
\frac{1}{\lambda}\n\end{cases}$   $\Rightarrow \begin{cases}\n\frac{1}{8\lambda} \\
\frac{1}{2\lambda} \\
\frac{1}{\lambda}\n\end{cases}$  $\lambda$  and  $\lambda$  $\lambda$  $\begin{cases}\n\lambda \neq 0 \\
1 - 8\lambda x = 0 \\
1 - 2\lambda y = 0\n\end{cases} \Rightarrow \begin{cases}\n\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
y = \frac{1}{2\lambda} \\
4x^2 + y^2 = 4\n\end{cases} \Rightarrow \begin{cases}\n\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
4(\frac{1}{8\lambda})^2 + (\frac{1}{2\lambda})^2 = 4\n\end{cases} \Rightarrow \begin{cases}\n\lambda \neq 0 \\
y = \frac{1}{2\lambda} \\
\frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 4\n\end{cases}$  $\begin{cases} 1 - 2\lambda y = 0 \\ 4x^2 + y^2 = 4 \end{cases}$   $\begin{cases} y = \frac{1}{2\lambda} \\ 4(\frac{1}{8\lambda})^2 + (\frac{1}{2\lambda})^2 = 4 \end{cases}$   $\begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \frac{5}{16\lambda^2} = 4 \end{cases}$   $\Rightarrow \begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \lambda^2 = \frac{5}{64} \end{cases}$   $\Rightarrow \begin{cases} \lambda \$  $\begin{cases} 4x^2 + y^2 = 4 \\ \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \frac{5}{16\lambda^2} = 4 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \lambda^2 = \frac{5}{64} \\ \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x = \pm \frac{1}{5}\sqrt{5} \\ y = \pm \frac{4}{5}\sqrt{5} \\ y = \pm \frac{4}{5}\sqrt{5} \end{cases}$ ; two  $\begin{cases}\n\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
y = \frac{1}{2\lambda} \\
\frac{5}{16\lambda^2} = 4\n\end{cases} \Rightarrow \begin{cases}\n\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
y = \frac{1}{2\lambda} \\
\lambda^2 = \frac{5}{64} \\
\end{cases} \Rightarrow \begin{cases}\n\lambda \neq 0 \\
x = \pm \frac{1}{5}\sqrt{5} \\
y = \pm \frac{4}{5}\sqrt{5} \\
\lambda = \pm \frac{1}{8}\sqrt{5}\n\end{cases};$ two<br>  $P_1\left(\frac{1}{5}\sqrt{5}, \frac{4}{5}\sqrt{$  $\lambda^2 = \frac{5}{64}$  $\lambda \neq 0$ 1-2 $\lambda y = 0$ <br>
4 $x^2 + y^2 = 4$ <br>  $\lambda \neq 0$ <br>  $x = \frac{1}{8\lambda}$ <br>  $y = \frac{1}{2\lambda}$ <br>  $y = \frac{1}{64}$ <br>  $\lambda^2 = \frac{5}{64}$ <br>  $\lambda = \pm \frac{1}{8}\sqrt{2}$  $1 - 8\lambda x = 0$ <br>  $1 - 2\lambda y = 0$   $\Rightarrow$   $\begin{cases} x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \end{cases}$ <br>  $4x^2 + y^2 = 4$   $\begin{cases} 4(\frac{1}{8\lambda})^2 + (\frac{1}{2\lambda})^2 = 0 \\ 4(\frac{1}{8\lambda})^2 + (\frac{1}{2\lambda})^2 = 0 \end{cases}$ <br>  $x = \frac{1}{8\lambda}$   $x = \frac{1}{8\lambda}$   $y = \frac{1}{2\lambda}$   $\Rightarrow$   $\begin{cases} x = \pm \\ y = \frac{1}{2\$  $y^2 = 4$   $\left(4\left(\frac{1}{8\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 0$ <br>  $0$   $\frac{1}{8\lambda}$   $\Rightarrow$   $\begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \end{cases}$   $\Rightarrow$   $\begin{cases} \lambda \neq 0 \\ x = \pm \frac{1}{5} \\ y = \pm \frac{4}{5} \\ \lambda = \pm \frac{1}{5} \end{cases}$ <br>  $= 4$   $\sqrt{5}, -\sqrt{5}$   $\Big)$ ,  $P_2\Big(-\frac{1}{2\lambda}\Big$  $\frac{1}{\lambda}$   $\frac{1}{\lambda}$   $\frac{1}{\lambda}$   $\frac{1}{\lambda}$   $\frac{1}{\lambda}$   $= \pm \frac{1}{5}\sqrt{5}$   $= \pm \frac{1}{5}\sqrt{5}$   $\frac{1}{\lambda}$   $\frac{1}{\lambda}$   $= \pm \frac{1}{5}\sqrt{5}$   $\frac{1}{\lambda}$   $\frac{1}{\lambda}$   $\frac{1}{\lambda}$   $\frac{1}{\lambda}$   $\frac{1}{\lambda}$   $\frac{1}{\lambda}$   $\frac{1}{\lambda}$   $\frac{1}{\lambda}$   $\frac{1$  $\Rightarrow \begin{cases} x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 4 \\ x = \pm \frac{1}{5}\sqrt{5} \\ y = \pm \frac{4}{5}\sqrt{5} \text{ ; two constrained } c \end{cases}$  $\left(\frac{1}{2\lambda}\right)^2 = 4$   $\left(\frac{1}{2\lambda}\right)^2 = 4$   $\frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 4$ <br>  $x = \pm \frac{1}{5}\sqrt{5}$ <br>  $y = \pm \frac{4}{5}\sqrt{5}$ ; two constrained c<br>  $\lambda = \pm \frac{1}{8}\sqrt{5}$  $\begin{array}{ll} \frac{1}{\lambda} & = 4 & \frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 4 \\ \neq 0 & \pm \frac{1}{5}\sqrt{5} \\ &= \pm \frac{4}{5}\sqrt{5} \\ &= \pm \frac{1}{8}\sqrt{5} \\ &= \pm \frac{1}{8}\sqrt{5} \\ -\frac{4}{5}\sqrt{5} \end{array}$ . The first point pre  $\lambda y = 0$ <br>  $y^2 = 4$ <br>  $\lambda y = \frac{1}{2\lambda}$ <br>  $\lambda \neq 0$ <br>  $\lambda \neq 0$ <br>  $x = \frac{1}{8\lambda}$ <br>  $x = \frac{1}{8\lambda}$ <br>  $y = \frac{1}{2\lambda}$ <br>  $x = \frac{1}{8\lambda}$ <br>  $y = \frac{1}{2\lambda}$  $+y^2 = 4$   $\left(4\left(\frac{1}{8\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2\right)$ <br>  $0$   $\frac{1}{8\lambda}$   $\Rightarrow$   $\begin{cases} \lambda \neq 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \end{cases}$   $\Rightarrow$   $\begin{cases} \lambda \neq 0 \\ x = 0 \\ y = 1 \end{cases}$  $\begin{cases}\n-y = 4 \\
y = \frac{1}{2\lambda}\n\end{cases}\n\Rightarrow\n\begin{cases}\n\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
y = \frac{1}{2\lambda}\n\end{cases}\n\Rightarrow\n\begin{cases}\n\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
y = \frac{1}{2\lambda}\n\end{cases}$  $\begin{cases}\n0 \\
\frac{1}{8\lambda} \\
\frac{1}{2\lambda} \\
=4\n\end{cases}$   $\Rightarrow$   $\begin{cases}\n\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
y = \frac{1}{2\lambda} \\
\lambda^2 = \frac{5}{64}\n\end{cases}$   $\Rightarrow$   $\begin{cases}\n\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
y = \frac{1}{2\lambda} \\
\lambda = \frac{1}{64}\n\end{cases}$  $\neq 0$   $= \frac{1}{8\lambda}$   $= \frac{1}{2\lambda}$   $y = z$   $y = z$   $y = z$   $y = z$   $y = z$  $x = \frac{1}{8\lambda}$ <br>  $y = \frac{1}{2\lambda}$   $\Rightarrow$   $\begin{cases} x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \lambda^2 = \frac{5}{64} \end{cases}$   $\Rightarrow$   $\begin{cases} x = 1 \\ y = 1 \\ \lambda = 1 \end{cases}$ <br>  $\begin{cases} 1, \sqrt{5} \\ 4, \sqrt{5} \end{cases}$   $\Rightarrow$   $\begin{cases} 1, \sqrt{5} \\ 2, \sqrt{5} \end{cases}$  $^{2} - \frac{5}{6}$  $1.75$  $5V^{\circ}$ .  $\frac{4}{5}$ ,  $\sqrt{5}$ ,  $\frac{1}{5}$  $5 \vee$  $\frac{1}{2}$ ,  $\sqrt{5}$  $8V^{\circ}$  $y^2 = 4$   $\left(4\left(\frac{1}{8\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2\right)$ <br>  $\frac{1}{\lambda}$   $\Rightarrow$   $\left\{\begin{array}{l}\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
y = \frac{1}{2\lambda}\end{array}\right\}$   $\Rightarrow$   $\left\{\begin{array}{l}\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
y = \frac{1}{2\lambda}\end{array}\right\}$  $-2\lambda y = 0 \Rightarrow \begin{cases} y = \frac{1}{2\lambda} \\ 4\left(\frac{1}{8\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 4 \end{cases}$ <br>  $\neq 0$ <br>  $= \frac{1}{8\lambda}$ <br>  $= \frac{1}{2\lambda}$ <br>  $y = \frac{1}{2\lambda}$ <br>  $y = \frac{1}{2\lambda}$ <br>  $\lambda^2 = \frac{5}{64}$ <br>  $\lambda = \pm \frac{1}{5}\sqrt{5}$ <br>  $y = \pm \frac{4}{5}\sqrt{5}$ <br>  $y = \pm \frac{4}{5}\sqrt{5}$  $\sqrt{5}$  $\sqrt{5}$ , two consumed  $\Rightarrow$   $\begin{cases} \frac{3}{y} = \frac{1}{2\lambda} \\ \frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 4 \end{cases}$ <br>  $\sqrt{\frac{5}{5}}$ ; two constrained critical points  $\begin{cases} \n\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
y = \frac{1}{2\lambda} \\
\frac{5}{16\lambda^2} = 4\n\end{cases} \Rightarrow \begin{cases} \n\lambda \neq 0 \\
x = \frac{1}{8\lambda} \\
y = \frac{1}{2\lambda} \\
\lambda^2 = \frac{5}{64}\n\end{cases} \Rightarrow \begin{cases} \n\lambda \neq 0 \\
x = \pm \frac{1}{5}\sqrt{5} \\
y = \pm \frac{4}{5}\sqrt{5}\n\end{cases}$ ; two constrained critical points<br>  $P_1\left(\frac{1$  $y = \frac{1}{2\lambda}$   $y = \frac{1}{2\lambda}$   $y = \frac{1}{2\lambda}$   $\lambda^2 = \frac{5}{64}$   $\lambda^2 = \frac{5}{64}$   $\lambda = \pm \frac{1}{8}\sqrt{5}$ <br>  $P_1\left(\frac{1}{5}\sqrt{5}, \frac{4}{5}\sqrt{5}\right)$ ,  $P_2\left(-\frac{1}{5}\sqrt{5}, -\frac{4}{5}\sqrt{5}\right)$ . The first point presents  $\lambda > 0$ , point of maximum, the sec  $\left(\frac{5}{16\lambda^2} - 4\right)\left(\lambda^2 - \frac{5}{64}\right)\left(\lambda = \pm \frac{1}{8}\sqrt{5}\right)$ <br>  $P_1\left(\frac{1}{5}\sqrt{5}, \frac{4}{5}\sqrt{5}\right), P_2\left(-\frac{1}{5}\sqrt{5}, -\frac{4}{5}\sqrt{5}\right)$ . The first point presents  $\lambda > 0$ , point of maximum, the second presents  $\lambda < 0$ , point of minimum.  $\frac{1}{16\lambda^2} = 4$   $\lambda^2 = \frac{2}{64}$   $\lambda = \pm \frac{1}{8}\sqrt{5}$ <br>  $\lambda_1\left(\frac{1}{5}\sqrt{5}, \frac{4}{5}\sqrt{5}\right)$ ,  $P_2\left(-\frac{1}{5}\sqrt{5}, -\frac{4}{5}\sqrt{5}\right)$ . The first point presents  $\lambda > 0$ , point of aximum, the second presents  $\lambda < 0$ , point of mimimum. II  $\left(\frac{1}{5}\sqrt{3}, \frac{4}{5}\sqrt{5}\right)$ ,  $12\left(-\frac{1}{5}\sqrt{3}, -\frac{1}{5}\sqrt{3}\right)$ . The first point presents  $\lambda > 0$ , point of<br>maximum, the second presents  $\lambda < 0$ , point of mimimum. We get the maximum<br> $f\left(\frac{1}{5}\sqrt{5}, \frac{4}{5}\sqrt{5}\right) = \sqrt{5}$   $\lambda < 0$ , point of mimimum. We get the maximum<br>he minimum  $f\left(-\frac{1}{5}\sqrt{5}, -\frac{4}{5}\sqrt{5}\right) = -\sqrt{5}$ .<br> $(x, y) = \begin{cases} \frac{(xy)^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$  is differentiable at t of mimimum. We get the maximum<br>  $\int_{1}^{1} f\left(-\frac{1}{5}\sqrt{5}, -\frac{4}{5}\sqrt{5}\right) = -\sqrt{5}.$ <br>  $\int_{\frac{(xy)^3}{x^2+y^2}}^{(\frac{(xy)^3}{x^2+y^2}}$  if  $(x, y) \neq (0, 0)$  is differentiable at<br>  $\int_{0}^{(\frac{h \cdot 0)^3}{x^2-y^2}}$  = 0  $3 \quad \blacksquare$  $\begin{aligned} f\Big(&-\frac{\overline{1}}{5}\sqrt{5},\ \frac{xy)^3}{x^2+y^2} \quad \text{if}\ (x,y) \ \text{if}\ (x,y) \end{aligned}$  $\neq (0,0)$  is different  $=(0,0)$  $f\left(\frac{1}{5}\sqrt{5}, \frac{4}{5}\sqrt{5}\right) = \sqrt{5}$  and the minimum  $f\left(-\frac{1}{5}\sqrt{5}, -\frac{4}{5}\sqrt{5}\right) = -\sqrt{5}$ .<br>
II M 3) Check if the function  $f(x, y) = \begin{cases} \frac{(xy)^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$  is differentiable at poi he function  $f(x, y) = \begin{cases} \frac{(xy)^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$  is differentiable as <br>  $\frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{(h \cdot 0)^3}{h^2 + 0^2} - 0}{h} = \lim_{h \to 0} \frac{0}{h} = 0;$ 

$$
(0,0)
$$
.

II M 3) Check if the function 
$$
f(x, y) = \begin{cases} \frac{x-y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}
$$
  
\n(0,0).  
\n
$$
f'_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{(h \cdot 0)^3}{h^2 + 0^2} - 0}{h} = \lim_{h \to 0} \frac{0}{h} = 0;
$$
\n
$$
f'_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{(0 \cdot h)^3}{0^2 + h^2} - 0}{h} = \lim_{h \to 0} \frac{0}{h} = 0.
$$
\nFunction *f* is differentiable at point (0,0) if  
\n
$$
\lim_{(x,y) \to (0,0)} \frac{f(x,y) - f(0,0) - (f'_x(0,0) \cdot x + f'_y(0,0) \cdot y)}{\sqrt{x^2 + y^2}} = 0, \text{ but}
$$
\n
$$
f(x,y) = f(0,0) - (f'_x(0,0) \cdot x + f'_y(0,0) \cdot y)
$$

Function f is differentiable at point  $(0, 0)$  if

$$
f'_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{\partial^2 h^2}{\partial x^2} - 0}{h} = \lim_{h \to 0} \frac{0}{h} = 0.
$$
  
Function f is differentiable at point (0,0) if  

$$
\lim_{(x,y) \to (0,0)} \frac{f(x,y) - f(0,0) - (f'_x(0,0) \cdot x + f'_y(0,0) \cdot y)}{\sqrt{x^2 + y^2}} = 0, \text{ but}
$$

$$
\lim_{(x,y) \to (0,0)} \frac{f(x,y) - f(0,0) - (f'_x(0,0) \cdot x + f'_y(0,0) \cdot y)}{\sqrt{x^2 + y^2}} =
$$

$$
\lim_{(x,y)\to(0,0)}\frac{f(x,y)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{\frac{(xy)^3}{x^2+y^2}}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\left(\frac{xy}{\sqrt{x^2+y^2}}\right)^3,
$$
\nwriting the limit in polar coordinates, it can be rewritten as:

writing the limit in polar coordinates, it can be rewritten as:

$$
\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\frac{1}{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \left(\frac{xy}{\sqrt{x^2+y^2}}\right) ,
$$
\n
$$
\text{writing the limit in polar coordinates, it can be rewritten as:}
$$
\n
$$
\lim_{\rho \to 0} \left( \frac{\rho \cdot \cos \theta \cdot \rho \cdot \sin \theta}{\sqrt{(\rho \cdot \cos \theta)^2 + (\rho \cdot \sin \theta)^2}} \right)^3 = \lim_{\rho \to 0} \left( \frac{\rho^2 \cdot \cos \theta \cdot \sin \theta}{\rho \sqrt{\cos^2 \theta + \sin^2 \theta}} \right)^3 =
$$
\n
$$
\lim_{\rho \to 0} \rho^3 (\cos \theta \cdot \sin \theta)^3 = 0. \text{ The convergence is uniformly because}
$$
\n
$$
|\rho^3 (\cos \theta \cdot \sin \theta)^3| = \rho^3 |\cos \theta \cdot \sin \theta|^3 \le \rho^3 (1/2)^3 = \rho^3 / 8; \text{ put}
$$

 $\lim_{\rho \to 0} \left( \frac{\rho \cdot \cos \theta \cdot \rho \cdot \sin \theta}{\sqrt{(\rho \cdot \cos \theta)^2 + (\rho \cdot \sin \theta)^2}} \right) = \lim_{\rho \to 0} \left( \frac{\rho \cdot \cos \theta \cdot \sin \theta}{\rho \sqrt{\cos^2 \theta + \sin^2 \theta}} \right) =$ <br>  $\lim_{\rho \to 0} \rho^3 (\cos \theta \cdot \sin \theta)^3 = 0.$  The convergence is uniformly because<br>  $|\rho^3 (\cos \theta \cdot \sin \theta)^3| = \rho^3 |\cos \theta \cdot \sin$  $\left(\sqrt{(\rho \cdot \cos \theta)^2 + (\rho \cdot \sin \theta)^2}\right)$   $\rho \to 0$   $\rho \to 0$ <br>  $\lim_{\rho \to 0} \rho^3 (\cos \theta \cdot \sin \theta)^3 = 0$ . The convergence is uniform  $|\rho^3 (\cos \theta \cdot \sin \theta)^3| = \rho^3 |\cos \theta \cdot \sin \theta|^3 \le \rho^3 (1/2)^3 =$ <br>  $\rho^3/8 < \epsilon \Leftrightarrow \rho^3 < 8\epsilon \Leftrightarrow \rho < 2 \sqrt[3]{\epsilon}$ .<br>
II M 4) Funct

 $\lim_{\rho \to 0} \rho^3 (\cos \theta \cdot \sin \theta)^3 = 0$ . The convergence is uniformly because<br>  $|\rho^3 (\cos \theta \cdot \sin \theta)^3| = \rho^3 |\cos \theta \cdot \sin \theta|^3 \le \rho^3 (1/2)^3 = \rho^3/8$ ; put<br>  $\rho^3/8 < \epsilon \Leftrightarrow \rho^3 < 8\epsilon \Leftrightarrow \rho < 2\sqrt[3]{\epsilon}$ .<br>
II M 4) Function  $f(x, y) = x^2 - y^2$  has direc  $\left[\rho^3(\cos\theta \cdot \sin\theta)^3\right] = \rho^3|\cos\theta \cdot \sin\theta|^3 \le \rho^3(1/2)^3 = \rho^3/8$ ; put<br>  $\sigma^3/8 < \epsilon \Leftrightarrow \rho^3 < 8\epsilon \Leftrightarrow \rho < 2\sqrt[3]{\epsilon}$ .<br>
II M 4) Function  $f(x, y) = x^2 - y^2$  has directional derivatives  $\mathcal{D}_v f(x_P, y_P) = 0$  and<br>  $\mathcal{D}_w f(x_P, y_P) = 0$ , wh  $\begin{aligned} &\langle \cos \theta \cdot \sin \theta \rangle^3 \Big| = \rho^3 |\cos \theta \cdot \sin \theta|^3 \le \rho^3 (1/2)^3 = \rho^3/8; \text{ put } \\ &\langle \epsilon \Leftrightarrow \rho^3 \langle \delta \epsilon \Leftrightarrow \rho \langle \epsilon \rangle^3 \langle \epsilon \rangle^3 \Big| \le \rho^3 (1/2)^3 = \rho^3/8; \text{ put } \\ &\text{4) Function } f(x, y) = x^2 - y^2 \text{ has directional derivatives } \mathcal{D}_v f(x_P, y_P) = 0 \text{ and } \\ &\langle (x_P, y_P) = 0 \text{, where } v \text{ is the unit vector } \$ II M 4) Function  $f(x, y) = x^2 - y^2$ <br>  $\mathcal{D}_w f(x_P, y_P) = 0$ , where v is the un<br>  $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ . Find the point  $(x, y)$ <br>  $\mathcal{D}_{v,w}^{(2)} f(x_P, y_P)$ . Function  $f(x, y) = x^2 - y^2$  has directional derivatives  $\mathcal{D}_v f(x_P, y_P) = 0$  and  $x_P, y_P) = 0$ , where v is the unit vector  $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$  and w is the unit vector  $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ . Find the point  $(x_P, y_P)$   $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ <br>  $\left(\frac{2}{v,w}f(x_P, y_P)\right)$ <br>
unction f is different  $D_w f(x_P, y_P) = 0$ , where v is there  $v$  is the  $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ . Find the point  $\left(\begin{array}{c} \frac{(2)}{2} & \frac{\sqrt{2}}{2} \\ \frac{(2)}{2} & \frac{\sqrt{2}}{2} \\ \frac{2}{2} & \frac{\sqrt{2}}{2} \end{array}\right)$ . Find the point  $\frac{(2)}{2}$  with  $f(x_P, y_P)$ .  $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ . Find the point  $(x_P, y_P)$  and calculate the directional derivative<br>  $\mathcal{D}_{v,w}^{(2)} f(x_P, y_P)$ .<br>
Function f is differentiable for any point  $(x, y)$  with gradient  $\nabla f(x, y) = (2x, -2y)$ ,<br>  $\mathcal{D}_v f(x_P, y_P)$ 

 $\mathcal{D}_{v,w}^{(2)} f(x_P, y_P)$ .<br>Function f is differentiable for any point  $(x, y)$  with gradient  $\nabla f(x, y) = (2x, -2y)$ .  $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ . Find the point  $(x_P, y_P)$  and calculate the directional derivative<br>  $D_{v,w}^{(2)} f(x_P, y_P)$ .<br>
Function f is differentiable for any point  $(x, y)$  with gradient  $\nabla f(x, y) = (2x, -2y)$ ,<br>  $\mathcal{D}_v f(x_P, y_P) = (2x$  $\left(\frac{Z}{2}, -\frac{\sqrt{2}}{2}\right)$ . Find the point  $(x_P, y_P)$  and calculate the directional derivative<br>  $f(x_P, y_P)$ .<br>
tion f is differentiable for any point  $(x, y)$  with gradient  $\nabla f(x, y) = (2x, -2y)$ ,<br>  $(x_P, y_P) = (2x_P, -2y_P) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\$ and Function f is differentiable for any point  $(x, y)$  with gradient  $\forall f(x, y) = (2x, -2y)$ ,<br>  $\mathcal{D}_v f(x_P, y_P) = (2x_P, -2y_P) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \sqrt{2}x_P - \sqrt{2}y_P = (x_P - y_P)\sqrt{2}$ <br>
and<br>  $\mathcal{D}_w f(x_P, y_P) = (2x_P, -2y_P) \cdot \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}$ 

Function *f* is an  
relation *f* is an  
equation *f* is an  

$$
\mathcal{D}_v f(x_P, y_P) = (2x_P, -2y_P) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \sqrt{2}x_P - \sqrt{2}y_P = (x_P - y_P)\sqrt{2}
$$
  
and  

$$
\mathcal{D}_w f(x_P, y_P) = (2x_P, -2y_P) \cdot \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \sqrt{2}x_P + \sqrt{2}y_P = (x_P + y_P)\sqrt{2}.
$$
  
Putting  $(x_P - y_P)\sqrt{2} = 0$  and  $(x_P + y_P)\sqrt{2} = 0$  it easily follows  $x_P = y_P = 0$ .  
Remember that 
$$
\mathcal{D}_{v,w}^{(2)} f(x_P, y_P) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \mathcal{H} f(x_P, y_P) \cdot \left(\begin{array}{c} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{array}\right)
$$
 and  

$$
\mathcal{H} f(x_P, y_P) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}
$$
; we get  

$$
\mathcal{D}_{v,w}^{(2)} f(x_P, y_P) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \cdot \left(\begin{array}{c} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{array}\right) =
$$
  

$$
\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot (\sqrt{2}, \sqrt{2}) = 2.
$$