# UNIVERSITA' DEGLI STUDI DI SIENA
# Scuola di Economia e Management
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# Quantitative Methods for Economic Applications - Mathematics for Economic Applications
# Task 4/7/2024

I M 1) Given the complex number $z = \dfrac{1-i}{1+i}$, write the complex number in goniometric form and calculate its cubic roots.

$z = \dfrac{1-i}{1+i} = \dfrac{1-i}{1+i} \cdot \dfrac{1-i}{1-i} = \dfrac{1-2i+i^2}{1-i^2} = \dfrac{-2i}{2} = -i$ (remember that $i^2 = -1$).

In goniometric form: $z = -i = cos\,\dfrac{3}{2}\pi + i\,sin\,\dfrac{3}{2}\pi$. For the cubic roots we apply the classical formula: $\sqrt[3]{z} = \sqrt[3]{cos\,\dfrac{3}{2}\pi + i\,sin\,\dfrac{3}{2}\pi} =$

$$cos\left(\frac{\frac{3}{2}\pi + 2k\pi}{3}\right) + i\,sin\left(\frac{\frac{3}{2}\pi + 2k\pi}{3}\right) = cos\left(\frac{\pi}{2} + \frac{2}{3}k\pi\right) + i\,sin\left(\frac{\pi}{2} + \frac{2}{3}k\pi\right)$$

$k = 0, 1, 2$. The three roots are:

$k = 0 \rightarrow z_0 = cos\left(\dfrac{\pi}{2}\right) + i\,sin\left(\dfrac{\pi}{2}\right) = i$;

$k = 1 \rightarrow z_1 = cos\left(\dfrac{7}{6}\pi\right) + i\,sin\left(\dfrac{7}{6}\pi\right) = -\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$;

$k = 2 \rightarrow z_2 = cos\left(\dfrac{11}{6}\pi\right) + i\,sin\left(\dfrac{11}{6}\pi\right) = \dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$.

I M 2) Given the matrix $\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, calculate its eigenvalues and study if the matrix $\mathbb{A}$ is diagonalizable or not.

At the first step we calculate the characteristic polynomial of matrix $\mathbb{A}$;

$$P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda-1 & -3 & 0 \\ 0 & -3 & \lambda-1 & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda-1 & -3 & 0 \\ -3 & \lambda-1 & 0 \\ 0 & 0 & \lambda \end{vmatrix} =$$

$$\lambda^2 \begin{vmatrix} \lambda-1 & -3 \\ -3 & \lambda-1 \end{vmatrix} = \lambda^2\left((\lambda-1)^2 - 9\right) = \lambda^2(\lambda^2 - 2\lambda - 8) = \lambda^2(\lambda-4)(\lambda+2).$$

Putting $P_{\mathbb{A}}(\lambda) = 0$ we find the four eigenvalues of matrix $\mathbb{A}$; if $\lambda^2 = 0$, we have the first two eigenvalues $\lambda_{1,2} = 0$; if $\lambda - 4 = 0$ it follows $\lambda_3 = 4$ and finally if $\lambda + 2 = 0$ it follows $\lambda_4 = -2$. The eigenvalue zero discloses algebraic multiplicity equal two, thus matrix $\mathbb{A}$ is a diagonalizable one if and only if the geometric multiplicity of the eigenvalue zero is two. To calculate such geometric multiplicity, we calculate the rank of the matrix $0 \cdot \mathbb{I} - \mathbb{A} = -\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; easily we can note that only one two by two minor of the matrix, the $\begin{vmatrix} -1 & -3 \\ -3 & -1 \end{vmatrix} = -8$ is different from zero, we conclude that the geometric multiplicity of the eigenvalue zero is two (the difference between the order of the matrix $\mathbb{A}$, 4, and the rank of the matrix $0 \cdot \mathbb{I} - \mathbb{A}$, 2). Matrix $\mathbb{A}$ is a digonalizable one.

I M 3) Given the linear application $F: \mathbb{R}^3 \to \mathbb{R}^3$, with $F(x_1, x_2, x_3) = (x_1 + x_2, x_1 + x_2 + x_3, x_2 + x_3)$.
Find the matrix associated with the linear application, calculate the dimentions of both, kernel and immage of $F$, and find a basis for the image.

If $F(x_1, x_2, x_3) = (x_1 + x_2, x_1 + x_2 + x_3, x_2 + x_3)$, the matrix associated to the linear application is $\mathbb{A}_F = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. The determinant of matrix $\mathbb{A}_F$ is $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} =$ $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 0 - 1 = -1 \neq 0$. Matrix $\mathbb{A}_F$ has rank three and we can conclude that the dimention of the Image of $F$ is 3, while the dimention of the Kernel is $dim(\mathbb{R}^3) - dim(Ima(F)) = 3 - 3 = 0$. Because the codomain of the linear application is $\mathbb{R}^3$ and dimention of the Image is again 3, easily follows that $Ima(F) = \mathbb{R}^3$ and a basis for the Image is the set $\mathcal{B}_{Ima(F)} = \mathcal{B}_{\mathbb{R}^3} = \{(1,0,0), (0,1,0), (0,0,1)\}$.

I M 4) Vector $V$ has coordinates $(1,1,1)$ respect the basis $\mathcal{B} = \{(0,0,1), (0,1,1), (1,1,1)\}$; find the coordinates of vector $V$ respect the basis $\mathcal{B}' = \{(1,1,1), (1,1,0), (1,0,0)\}$. Is the set $\mathcal{C} = \{(1,1,1), (1,0,1), (0,1,0)\}$ a basis for the vector space $\mathbb{R}^3$?

If vector $V$ has coordinates $(1,1,1)$ respect the basis $\mathcal{B}$ and coordinates $(\alpha, \beta, \chi)$ respect the basis $\mathcal{B}'$, $V = 1 \cdot (0,0,1) + 1 \cdot (0,1,1) + 1 \cdot (1,1,1) =$ $(1,2,3)$ and at the same time $V = \alpha \cdot (1,1,1) + \beta \cdot (1,1,0) + \chi \cdot (1,0,0) =$ $(\alpha + \beta + \chi, \alpha + \beta, \alpha)$. Putting $(1,2,3) = (\alpha + \beta + \chi, \alpha + \beta, \alpha)$ it easily follows $\alpha = 3$ and $\beta = \chi = -1$. For the second part of the exercise, note that $(1,0,1) + (0,1,0) = (1,1,1)$, thus the set $\mathcal{C}$ is a set of three linear dependent vectors belonging to $\mathbb{R}^3$ and easily we conclude that $\mathcal{C}$ isn't a basis for the vector space $\mathbb{R}^3$.

II M 1) Given the equation $x \cdot e^{x+y-z^2} - z^2 \cdot e^{x+y+z} = 0$ satisfied at the point $P(1, 0, -1)$; verify that with it an implicit function $(x, y) \mapsto z(x, y)$ can be defined and then calculate, for this implicit function, the partial derivatives $z'_x$ and $z'_y$.

Consider the function $f: \mathbb{R}^3 \to \mathbb{R}$ with $f(x, y, z) = x \cdot e^{x+y-z^2} - z^2 \cdot e^{x+y+z}$.
$f(P) = 1 \cdot e^0 - 1 \cdot e^0 = 0$, the partial derivative of $f$ respect the variable $z$ is
$f'_z = x \cdot e^{x+y-z^2} \cdot (-2z) - 2z \cdot e^{x+y+z} - z^2 \cdot e^{x+y+z} =$
$-z\left(2x \cdot e^{x+y-z^2} + (2+z)e^{x+y+z}\right)$; on point $P(1, 0, -1)$ the partial derivative has value $f'_z(P) = 1 \cdot (2 \cdot e^0 + 1 \cdot e^0) = 3 \neq 0$; the proposed equation defines in a neighbourhood of point $P$ an implicit function $(x, y) \mapsto z(x, y)$. To calculate the partial derivatives $z'_x$ and $z'_y$ we must firstly calculate the two partial derivatives $f'_x$ and $f'_y$:
$f'_x = 1 \cdot e^{x+y-z^2} + x \cdot e^{x+y-z^2} - z^2 \cdot e^{x+y+z} = (1+x) \cdot e^{x+y-z^2} - z^2 \cdot e^{x+y+z}$;
$f'_y = x \cdot e^{x+y-z^2} - z^2 \cdot e^{x+y+z} = f(x, y, z)$;

with $f'_x(P) = 2 \cdot e^0 - 1 \cdot e^0 = 1$ and $f'_y(P) = f(P) = 0$. The two partial derivatives of function $z$ are $z'_x(1,0) = -\dfrac{f'_x(P)}{f'_z(P)} = -\dfrac{1}{3}$ and $z'_y(1,0) = -\dfrac{f'_y(P)}{f'_z(P)} = -\dfrac{0}{3} = 0$.

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x+y \\ \text{u.c.: } 4x^2+y^2 \le 4 \end{cases}$.

The function $f$ is a polynomial, continuos function, the admissible region is an ellipse with center $(0,0)$ and axises of lengths equal to $2$ and $4$, a bounded and closed set, therefore $f$ presents absolute maximum and minimum in the admissible region. The Lagrangian function is

$\mathcal{L}(x,y,\lambda) = x + y - \lambda(4x^2+y^2-4)$ with

$\nabla\mathcal{L} = (1 - 8\lambda x, 1 - 2\lambda y, -(4x^2+y^2-4))$.

$I^\circ\ CASE$ ($free\ optimization$):

$\begin{cases} \lambda = 0 \\ 1 = 0 \\ 1 = 0 \\ 4x^2+y^2 \le 4 \end{cases}$. System impossible.

$II^\circ\ CASE$ ($constrained\ optimization$):

$$\begin{cases} \lambda \ne 0 \\ 1 - 8\lambda x = 0 \\ 1 - 2\lambda y = 0 \\ 4x^2+y^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda \ne 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ 4\left(\frac{1}{8\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda \ne 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 4 \end{cases} \Rightarrow$$

$$\begin{cases} \lambda \ne 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \frac{5}{16\lambda^2} = 4 \end{cases} \Rightarrow \begin{cases} \lambda \ne 0 \\ x = \frac{1}{8\lambda} \\ y = \frac{1}{2\lambda} \\ \lambda^2 = \frac{5}{64} \end{cases} \Rightarrow \begin{cases} \lambda \ne 0 \\ x = \pm\frac{1}{5}\sqrt{5} \\ y = \pm\frac{4}{5}\sqrt{5} \\ \lambda = \pm\frac{1}{8}\sqrt{5} \end{cases}$$ ; two constrained critical points

$P_1\left(\frac{1}{5}\sqrt{5}, \frac{4}{5}\sqrt{5}\right)$, $P_2\left(-\frac{1}{5}\sqrt{5}, -\frac{4}{5}\sqrt{5}\right)$. The first point presents $\lambda > 0$, point of maximum, the second presents $\lambda < 0$, point of mimimum. We get the maximum $f\left(\frac{1}{5}\sqrt{5}, \frac{4}{5}\sqrt{5}\right) = \sqrt{5}$ and the minimum $f\left(-\frac{1}{5}\sqrt{5}, -\frac{4}{5}\sqrt{5}\right) = -\sqrt{5}$.

II M 3) Check if the function $f(x,y) = \begin{cases} \frac{(xy)^3}{x^2+y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is differentiable at point $(0,0)$.

$$f'_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{(h\cdot 0)^3}{h^2+0^2} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0;$$

$$f'_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{(0\cdot h)^3}{0^2+h^2} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Function $f$ is differentiable at point $(0,0)$ if

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \left(f'_x(0,0)\cdot x + f'_y(0,0)\cdot y\right)}{\sqrt{x^2+y^2}} = 0, \text{ but}$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \left(f'_x(0,0)\cdot x + f'_y(0,0)\cdot y\right)}{\sqrt{x^2+y^2}} =$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\frac{(xy)^3}{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \left(\frac{xy}{\sqrt{x^2+y^2}}\right)^3,$$
writing the limit in polar coordinates, it can be rewritten as:

$$\lim_{\rho\to 0} \left(\frac{\rho\cdot cos\,\theta\cdot\rho\cdot sin\,\theta}{\sqrt{(\rho\cdot cos\,\theta)^2+(\rho\cdot sin\,\theta)^2}}\right)^3 = \lim_{\rho\to 0}\left(\frac{\rho^2\cdot cos\,\theta\cdot sin\,\theta}{\rho\sqrt{cos^2\theta+sin^2\theta}}\right)^3 =$$

$\lim_{\rho\to 0}\rho^3(cos\,\theta\cdot sin\,\theta)^3 = 0$. The convergence is uniformly because
$\left|\rho^3(cos\,\theta\cdot sin\,\theta)^3\right| = \rho^3|cos\,\theta\cdot sin\,\theta|^3 \le \rho^3(1/2)^3 = \rho^3/8$; put
$\rho^3/8 < \epsilon \Leftrightarrow \rho^3 < 8\epsilon \Leftrightarrow \rho < 2\sqrt[3]{\epsilon}$.

II M 4) Function $f(x,y) = x^2 - y^2$ has directional derivatives $\mathcal{D}_v f(x_P, y_P) = 0$ and $\mathcal{D}_w f(x_P, y_P) = 0$, where $v$ is the unit vector $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $w$ is the unit vector $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. Find the point $(x_P, y_P)$ and calculate the directional derivative $\mathcal{D}^{(2)}_{v,w} f(x_P, y_P)$.

Function $f$ is differentiable for any point $(x,y)$ with gradient $\nabla f(x,y) = (2x, -2y)$,

$$\mathcal{D}_v f(x_P, y_P) = (2x_P, -2y_P)\cdot\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \sqrt{2}x_P - \sqrt{2}y_P = (x_P - y_P)\sqrt{2}$$

and

$$\mathcal{D}_w f(x_P, y_P) = (2x_P, -2y_P)\cdot\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \sqrt{2}x_P + \sqrt{2}y_P = (x_P + y_P)\sqrt{2}.$$

Putting $(x_P - y_P)\sqrt{2} = 0$ and $(x_P + y_P)\sqrt{2} = 0$ it easily follows $x_P = y_P = 0$.

Remember that $\mathcal{D}^{(2)}_{v,w} f(x_P, y_P) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\cdot\mathcal{H}f(x_P, y_P)\cdot\begin{pmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{pmatrix}$ and $\mathcal{H}f(x_P, y_P) = \begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix}$; we get

$$\mathcal{D}^{(2)}_{v,w} f(x_P, y_P) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\cdot\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix}\cdot\begin{pmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{pmatrix} =$$

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\cdot\left(\sqrt{2}, \sqrt{2}\right) = 2.$$