UNIVERSITA' DEGLI STUDI DI SIENA
Scuola di Economia e Management
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Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 26/8/2024

I M 1) Given the complex number $z = \dfrac{1}{1+i} - \dfrac{1}{1-i}$, find the real part and the imaginary part of z and calculate its cubic roots.

$z = \dfrac{1}{1+i} - \dfrac{1}{1-i} = \dfrac{1-i-(1+i)}{(1+i)(1-i)} = \dfrac{-2i}{1-i^2} = \dfrac{-2i}{2} = -i$ (remember that $i^2 = -1$). $Re(z) = 0$, $Im(z) = -1$. In goniometric form:

$z = -i = cos\,\dfrac{3}{2}\pi + i\,sin\,\dfrac{3}{2}\pi$.

For the cubic roots we apply the classical formula: $\sqrt[3]{z} = \sqrt[3]{cos\,\dfrac{3}{2}\pi + i\,sin\,\dfrac{3}{2}\pi} =$

$cos\left(\dfrac{3\pi/2 + 2k\pi}{3}\right) + i\,sin\left(\dfrac{3\pi/2 + 2k\pi}{3}\right) = cos\left(\dfrac{\pi}{2} + \dfrac{2}{3}k\pi\right) + i\,sin\left(\dfrac{\pi}{2} + \dfrac{2}{3}k\pi\right)$

$k = 0, 1, 2$. The three roots are:

$k = 0 \rightarrow z_0 = cos\left(\dfrac{\pi}{2}\right) + i\,sin\left(\dfrac{\pi}{2}\right) = i$;

$k = 1 \rightarrow z_1 = cos\left(\dfrac{7}{6}\pi\right) + i\,sin\left(\dfrac{7}{6}\pi\right) = -\dfrac{1}{2}\left(\sqrt{3} + i\right) = -\overline{z_2}$;

$k = 2 \rightarrow z_2 = cos\left(\dfrac{11}{6}\pi\right) + i\,sin\left(\dfrac{11}{6}\pi\right) = \dfrac{1}{2}\left(\sqrt{3} - i\right) = -\overline{z_1}$.

I M 2) The matrix $\mathbb{A} = \begin{bmatrix} 8 & k \\ k & 8 \end{bmatrix}$ has an eigenvalue $\lambda = 4$, where k is a positive constant. Calculate the value of k and find the matrix $\mathbb{B}$ that diagonalizes matrix $\mathbb{A}$.

If $\lambda = 4$ is an eigenvalue of matrix $\mathbb{A}$, the determinant of the matrix $4 \cdot \mathbb{I} - \mathbb{A}$ is zero.

$|4 \cdot \mathbb{I} - \mathbb{A}| = \begin{vmatrix} -4 & -k \\ -k & -4 \end{vmatrix} = 16 - k^2$, put $16 - k^2 = 0$ we get $k^2 = 16 \Rightarrow k = 4$, because k is a positive constant. To find matrix $\mathbb{B}$ that diagonalizes matrix $\mathbb{A}$, we must start with the calculus of the second eigenvalue of the matrix $\mathbb{A}$, for our goal we calculate the characteristic polynomial of matrix $\mathbb{A}$;

$P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda - 8 & -4 \\ -4 & \lambda - 8 \end{vmatrix} = (\lambda - 8)^2 - 16$. Putting $P_{\mathbb{A}}(\lambda) = 0$ we have $(\lambda - 8)^2 = 16 \Rightarrow \lambda = 8 \pm 4$; $\lambda_1 = 4$ and $\lambda_2 = 12$.

Alternative Method: remember that the sum of all eigevalues of a matrix is equal to the matrix's trace, the sum of the elements in the principal diagonal of the matrix; thus for matrix $\mathbb{A}$, $\lambda_1 + \lambda_2 = 8 + 8$, by $\lambda_1 = 4$ trivially follows $\lambda_2 = 12$

Now we calculate the eigenvectors connected with any eigenvalue of matrix $\mathbb{A}$:

1. for $\lambda_1 = 4$ an eigenvector connected with λ_1 is a vector (x, y) such that

$(4\mathbb{I} - \mathbb{A}) \cdot (x, y) = (0, 0) \Leftrightarrow \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$

$\begin{pmatrix} -4x-4y \\ -4x-4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x+y=0$; any eigenvector connected with λ_1 is a vector $(x, -x)$, for instance $(1, -1)$;

2. for $\lambda_2 = 12$ an eigenvector connected with λ_2 is a vector (x,y) such that $(12\mathbb{I} - \mathbb{A}) \cdot (x,y) = (0,0) \Leftrightarrow \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$

$\begin{pmatrix} 4x-4y \\ -4x+4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x-y=0$; any eigenvector connected with λ_2 is a vector (x,x), for instance $(1,1)$.

Matrix $\mathbb{B}$ that diagonalizes matrix $\mathbb{A}$ is $\mathbb{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

To check the result, we calculate the product $\mathbb{B}^{-1} \cdot \mathbb{A} \cdot \mathbb{B}$.

$\mathbb{B}^{-1} \cdot \mathbb{A} \cdot \mathbb{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} =$

$\frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{T} \cdot \begin{bmatrix} 4 & 12 \\ -4 & 12 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 12 \\ -4 & 12 \end{bmatrix} =$

$\frac{1}{2} \begin{bmatrix} 8 & 0 \\ 0 & 24 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 12 \end{bmatrix}$, a diagonal matrix.

I M 3) Given the linear function $F: \mathbb{R}^3 \to \mathbb{R}^4$, we know that for function F:

1. $F(1,0,0) = (0,0,0,0)$;
2. $F(1,1,0) = (1,0,0,0)$;
3. $F(1,1,1) = (1,1,0,0)$.

Calculate the dimention of the kernel and the dimention of the image of F and find a basis for both, kernel and image.

By 2. and 3. linear indipendent vectors $(1,0,0,0)$ and $(1,1,0,0)$ belong to the image of function F, thus the dimention of the image is greater or equal 2; at the same time by 1. vector $(1,0,0)$ belongs to the kernel of function F, thus the dimention of the kernel is greater or equal then 1, but by the dimention Theorem, $dim(Ima(F)) + dim(Ker(F)) = dim(\mathbb{R}^3) = 3$, thus $dim(Ima(F)) = 2$ and $dim(Ker(F)) = 1$.

For the basis of both sets, by the linear independency of vectors $(1,0,0,0)$ and $(1,1,0,0)$ we have: $\mathcal{B}_{Ima(F)} = \{(1,0,0,0),(1,1,0,0)\}$ and $\mathcal{B}_{Ker(F)} = \{(1,0,0)\}$.

Alternative Solution: by 1., 2. and 3. we can calculate the matrix $\underset{4\times 3}{\mathbb{A}_F}$ associated at function F, $\mathbb{A}_F = \begin{bmatrix} \alpha & \beta & \chi \\ \delta & F & \epsilon \\ \eta & \gamma & \iota \\ \kappa & \mu & \nu \end{bmatrix}$.

From 1. $F(1,0,0) = \begin{bmatrix} \alpha & \beta & \chi \\ \delta & F & \epsilon \\ \eta & \gamma & \iota \\ \kappa & \mu & \nu \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \delta \\ \eta \\ \kappa \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$\alpha = \delta = \eta = \kappa = 0$; $\mathbb{A}_F = \begin{bmatrix} 0 & \beta & \chi \\ 0 & F & \epsilon \\ 0 & \gamma & \iota \\ 0 & \mu & \nu \end{bmatrix}$.

From 2. $F(1,1,0) = \begin{bmatrix} 0 & \beta & \chi \\ 0 & F & \epsilon \\ 0 & \gamma & \iota \\ 0 & \mu & \nu \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta \\ F \\ \gamma \\ \mu \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$\beta = 1$ and $F = \gamma = \mu = 0$; $\mathbb{A}_F = \begin{bmatrix} 0 & 1 & \chi \\ 0 & 0 & \epsilon \\ 0 & 0 & \iota \\ 0 & 0 & \nu \end{bmatrix}$.

From 3. $F(1,1,1) = \begin{bmatrix} 0 & 1 & \chi \\ 0 & 0 & \epsilon \\ 0 & 0 & \iota \\ 0 & 0 & \nu \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+\chi \\ \epsilon \\ \iota \\ \nu \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$\epsilon = 1$ and $\chi = \iota = \nu = 0$; $\mathbb{A}_F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Trivially we can notice that matrix $\mathbb{A}_F$ shows rank equal 2, thus the dimention of the image of function F is 2 and the dimention of the kernel is $dim(Ker(F)) = dim(\mathbb{R}^3) - dim(Ima(F)) = 3 - 2 = 1$.
For the basis of the image, by the linear independency of the second and the third columns of matrix $\mathbb{A}_F$, we can take for the basis of image the set: $\mathcal{B}_{Ima(F)} = \{(1,0,0,0),(0,1,0,0)\}$. For the kernel we know that a vector (x,y,z) belongs in the kernel of function F if and only if $F(x,y,z) = (0,0,0,0)$, but $F(x,y,z) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow y = z = 0$. Thus a generic vector that belongs in the kernel is a vector $(x,0,0) = x(1,0,0)$, and a basis for the kernel is the set $\mathcal{B}_{Ker(F)} = \{(1,0,0)\}$.

I M 4) The matrix $\mathbb{A} = \begin{bmatrix} \alpha & 2 & 3 \\ 0 & \alpha & 2 \\ 0 & 0 & \alpha \end{bmatrix}$ has determinant equal 8, where α is a positive constant.
Calculate the value of α and find matrix $\mathbb{A}^{-1}$, the inverse matrix of matrix $\mathbb{A}$.
Matrix $\mathbb{A}$ is an up-triangular matrix, following that the determinat of $\mathbb{A}$ is the product of its elements in the principal diagonal: $|\mathbb{A}| = \alpha^3$, put $\alpha^3 = 8$ we get $\alpha = 2$;
$\mathbb{A} = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. Remember that the inverse matrix of $\mathbb{A}$ is the matrix
$\mathbb{A}^{-1} = \frac{1}{|\mathbb{A}|}(Adj(\mathbb{A}))^T$, where $Adj(\mathbb{A})$ is the adjoin matrix of $\mathbb{A}$.

$$\mathbb{A}^{-1} = \frac{1}{8}\begin{bmatrix} \begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^T = \frac{1}{8}\begin{bmatrix} 4 & 0 & 0 \\ -4 & 4 & 0 \\ -2 & -4 & 4 \end{bmatrix}^T =$$

$$\frac{1}{8}\begin{bmatrix}4 & -4 & -2\\ 0 & 4 & -4\\ 0 & 0 & 4\end{bmatrix} = \begin{bmatrix}1/2 & -1/2 & -1/4\\ 0 & 1/2 & -1/2\\ 0 & 0 & 1/2\end{bmatrix}.$$

To check the result, we calculate the product $\mathbb{A}^{-1}\cdot\mathbb{A}$.

$\mathbb{A}^{-1}\cdot\mathbb{A} = \begin{bmatrix}1/2 & -1/2 & -1/4\\ 0 & 1/2 & -1/2\\ 0 & 0 & 1/2\end{bmatrix}\cdot\begin{bmatrix}2 & 2 & 3\\ 0 & 2 & 2\\ 0 & 0 & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$, an identity matrix.

II M 1) Solve the problem $\begin{cases}\text{Max/min } f(x,y) = 4x+4y\\ \text{u.c.: } x^2+y^2\le 4\end{cases}$.

The function f is an affine continuos function, the admissible region is a circle with center $(0,0)$ and radius equal 2, a bounded and closed set, therefore f presents absolute maximum and minimum in the admissible region. The Lagrangian function is
$\mathcal{L}(x,y,\lambda) = 4x+4y-\lambda(x^2+y^2-4)$ with
$\nabla\mathcal{L} = (4-2\lambda x, 4-2\lambda y, -(x^2+y^2-4))$.

$I^\circ\ CASE\ (free\ optimization)$:

$\begin{cases}\lambda = 0\\ 4 = 0\\ 4 = 0\\ x^2+y^2\le 4\end{cases}$. System impossible.

$II^\circ\ CASE\ (constrained\ optimization)$:

$\begin{cases}\lambda\ne 0\\ 4-2\lambda x = 0\\ 4-2\lambda y = 0\\ x^2+y^2 = 4\end{cases} \Rightarrow \begin{cases}\lambda\ne 0\\ x = \frac{2}{\lambda}\\ y = \frac{2}{\lambda}\\ \left(\frac{2}{\lambda}\right)^2+\left(\frac{2}{\lambda}\right)^2 = 4\end{cases} \Rightarrow \begin{cases}\lambda\ne 0\\ x = \frac{2}{\lambda}\\ y = \frac{2}{\lambda}\\ \frac{8}{\lambda^2} = 4\end{cases} \Rightarrow \begin{cases}\lambda\ne 0\\ x = \frac{2}{\lambda}\\ y = \frac{2}{\lambda}\\ \lambda^2 = 2\end{cases} \Rightarrow \begin{cases}\lambda\ne 0\\ x = \pm\sqrt{2}\\ y = \pm\sqrt{2}\\ \lambda = \pm\sqrt{2}\end{cases}$;

two constrained critical points $P_1\left(\sqrt{2},\sqrt{2}\right)$, $P_2\left(-\sqrt{2}, -\sqrt{2}\right)$. The first point presents $\lambda > 0$, point of maximum, the second presents $\lambda < 0$, point of mimimum. We get the maximum $f\left(\sqrt{2},\sqrt{2}\right) = 8\sqrt{2}$ and the minimum
$f\left(-\sqrt{2}, -\sqrt{2}\right) = -8\sqrt{2}$.

Alternative Solution: The function f is an affine continuos function, the admissible region is a circle with center $(0,0)$ and radius equal 2, a bounded and closed set, this implies that the point of maximum and the point of minimum must be found on the border of the admissible region. We write the Lagrangian function of the problem as
$\mathcal{L}(x,y,\lambda) = 4x+4y-\lambda(x^2+y^2-4)$ with
$\nabla\mathcal{L} = (4-2\lambda x, 4-2\lambda y, -(x^2+y^2-4))$.

FOC:

$\begin{cases}4-2\lambda x = 0\\ 4-2\lambda y = 0\\ x^2+y^2 = 4\end{cases} \Rightarrow \begin{cases}x = \frac{2}{\lambda}\\ y = \frac{2}{\lambda}\\ \left(\frac{2}{\lambda}\right)^2+\left(\frac{2}{\lambda}\right)^2 = 4\end{cases} \Rightarrow \begin{cases}x = \frac{2}{\lambda}\\ y = \frac{2}{\lambda}\\ \frac{8}{\lambda^2} = 4\end{cases} \Rightarrow \begin{cases}x = \pm\sqrt{2}\\ y = \pm\sqrt{2}\\ \lambda = \pm\sqrt{2}\end{cases}$; two constraint critical points $P_{1,2} = \left(\pm\sqrt{2}, \pm\sqrt{2}\right)$.

SOC:

$\overline{\mathcal{H}} = \begin{bmatrix}0 & -2x & -2y\\ -2x & -2\lambda & 0\\ -2y & 0 & -2\lambda\end{bmatrix}$, with $|\overline{\mathcal{H}}| = \begin{vmatrix}0 & -2x & -2y\\ -2x & -2\lambda & 0\\ -2y & 0 & -2\lambda\end{vmatrix} =$

$$-2\lambda \cdot \begin{vmatrix} 0 & -2x \\ -2x & -2\lambda \end{vmatrix} - 2y \cdot \begin{vmatrix} -2x & -2y \\ -2\lambda & 0 \end{vmatrix} = 8\lambda x^2 + 8\lambda y^2 = 8\lambda(x^2+y^2).$$

$\left|\overline{\mathcal{H}}(P_1)\right| = 32\sqrt{2} > 0$, P_1 point of maximum with $f(P_1) = 8\sqrt{2}$,
$\left|\overline{\mathcal{H}}(P_2)\right| = -32\sqrt{2} < 0$, P_2 point of minimum with $f(P_2) = -8\sqrt{2}$.

II M 2) Given the equation $\left(x^8+y^8+z^8\right) - \left(x^6+y^6+z^6\right) = 0$ satisfied at the point $P(1,1,-1)$; verify that with it an implicit function $(x,y) \mapsto z(x,y)$ can be defined and then calculate, for this implicit function, the partial derivatives z'_x and z'_y.
Consider the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ with $f(x,y,z) = (x^8+y^8+z^8) - (x^6+y^6+z^6)$.
$f(P) = 3-3 = 0$ and f is differentiable at any point (x,y,z), the partial derivative of f respect the variable z is $f'_z = 8z^7 - 6z^5$; at point $P(1,1,-1)$ the partial derivative has value $f'_z(P) = -8+6 = -2 \neq 0$; the proposed equation defines in a neighbourhood of point P an implicit function $(x,y) \mapsto z(x,y)$. To calculate the partial derivatives z'_x and z'_y we must firstly calculate the two partial derivatives f'_x and f'_y: $f'_x = 8x^7 - 6x^5$; $f'_y = 8y^7 - 6y^5$; with $f'_x(P) = f'_y(P) = 2$. The two partial derivatives of function z are $z'_x(1,1) = -\dfrac{f'_x(P)}{f'_z(P)} = 1$ and $z'_y(1,1) = -\dfrac{f'_y(P)}{f'_z(P)} = 1$.

II M 3) Given the function $f(x,y) = cos(x+y)$, calculate the second order directional derivative $\mathcal{D}^{(2)}_{v,w}f(0,0)$; where v is the unit vector $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$ and w is the unit vector $\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$.
Function f is differentiable at any point (x,y) with gradient vector $\nabla f(x,y) = (-sen(x+y), -sen(x+y))$ and hessian matrix
$\mathcal{H}f(x,y) = \begin{bmatrix} -cos(x+y) & -cos(x+y) \\ -cos(x+y) & -cos(x+y) \end{bmatrix}$ with $\mathcal{H}f(0,0) = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$.

$$\mathcal{D}^{(2)}_{v,w}f(0,0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \mathcal{H}f(0,0) \cdot \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} =$$

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \cdot \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \left(\sqrt{2}, \sqrt{2}\right) = 2.$$

II M 4) Calculate the partial derivatives of function $f(x,y,z,w) = e^{x+2y} - 3zw^3$.

$$f'_x = e^{x+2y} \qquad f'_y = 2e^{x+2y} \qquad f'_z = -3w^3 \qquad f'_w = -9zw^2.$$