UNIVERSITA' DEGLI STUDI DI SIENA
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Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 10/9/2024

I M 1) Given the complex number $z = \frac{1}{i-1} + \frac{1}{i+1}$, find the real part and the imaginary part of z and calculate its square roots.
$z = \frac{1}{i-1} + \frac{1}{i+1} = \frac{i+1+i-1}{(i-1)(i+1)} = \frac{2i}{i^2-1} = -\frac{2i}{2} = -i$ (remember that $i^2 = -1$). $Re(z) = 0$, $Im(z) = -1$. In goniometric form:
$z = -i = cos\,\frac{3}{2}\pi + i\,sin\,\frac{3}{2}\pi$.

For the square roots we apply the classical formula: $\sqrt{z} = \sqrt{cos\,\frac{3}{2}\pi + i\,sin\,\frac{3}{2}\pi} =$
$cos\left(\frac{3\pi/2 + 2k\pi}{2}\right) + i\,sin\left(\frac{3\pi/2 + 2k\pi}{2}\right) = cos\left(\frac{3}{4}\pi + k\pi\right) + i\,sin\left(\frac{3}{4}\pi + k\pi\right)$
$k = 0, 1$. The two roots are:
$k = 0 \rightarrow\ z_0 = cos\left(\frac{3}{4}\pi\right) + i\,sin\left(\frac{3}{4}\pi\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$;
$k = 1 \rightarrow\ z_1 = cos\left(\frac{7}{4}\pi\right) + i\,sin\left(\frac{7}{4}\pi\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = -z_0$.

I M 2) The matrix $\mathbb{A} = \begin{bmatrix} k & 8 \\ 8 & k \end{bmatrix}$ has an eigenvalue $\lambda = 4$, where k is a positive constant. Calculate the value of k and find the matrix $\mathbb{B}$ that diagonalizes matrix $\mathbb{A}$.
If $\lambda = 4$ is an eigenvalue of matrix $\mathbb{A}$, the determinant of the matrix $4 \cdot \mathbb{I} - \mathbb{A}$ is zero.
$|4 \cdot \mathbb{I} - \mathbb{A}| = \begin{vmatrix} 4-k & -8 \\ -8 & 4-k \end{vmatrix} = (4-k)^2 - 64 = (k-4)^2 - 64$, put
$(k-4)^2 - 64 = 0$ we get $(k-4)^2 = 64 \Rightarrow k = 4 \pm \sqrt{64} = 4 \pm 8 \Rightarrow k = 12$, because k is a positive constant. To find matrix $\mathbb{B}$ that diagonalizes matrix $\mathbb{A}$, we must start with the calculus of the second eigenvalue of the matrix $\mathbb{A}$, for our goal we calculate the characteristic polynomial of matrix $\mathbb{A}$;
$P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda - 12 & -8 \\ -8 & \lambda - 12 \end{vmatrix} = (\lambda - 12)^2 - 64$. Putting $P_{\mathbb{A}}(\lambda) = 0$ we have $(\lambda - 12)^2 = 64 \Rightarrow \lambda = 12 \pm 8$; $\lambda_1 = 4$ and $\lambda_2 = 20$.
Alternative Method: remember that the sum of all eigevalues of a matrix is equal to the matrix's trace, the sum of the elements in the principal diagonal of the matrix; thus for matrix $\mathbb{A}$, $\lambda_1 + \lambda_2 = 12 + 12$, by $\lambda_1 = 4$ trivially follows $\lambda_2 = 20$.
Now we calculate the eigenvectors connected with any eigenvalue of matrix $\mathbb{A}$:
1. for $\lambda_1 = 4$ an eigenvector connected with λ_1 is a vector (x, y) such that
$(4\mathbb{I} - \mathbb{A}) \cdot (x, y) = (0, 0) \Leftrightarrow \begin{bmatrix} -8 & -8 \\ -8 & -8 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$

$\begin{pmatrix} -8x-8y \\ -8x-8y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x+y=0$; any eigenvector connected with λ_1 is a vector $(x, -x)$, for instance $(1, -1)$;

2. for $\lambda_2 = 20$ an eigenvector connected with λ_2 is a vector (x,y) such that
$(20\mathbb{I} - \mathbb{A}) \cdot (x,y) = (0,0) \Leftrightarrow \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$
$\begin{pmatrix} 8x-8y \\ -8x+8y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x-y=0$; any eigenvector connected with λ_2 is a vector (x,x), for instance $(1,1)$.

Matrix $\mathbb{B}$ that diagonalizes matrix $\mathbb{A}$ is $\mathbb{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

To check the result, we calculate the product $\mathbb{B}^{-1} \cdot \mathbb{A} \cdot \mathbb{B}$.

$$\mathbb{B}^{-1} \cdot \mathbb{A} \cdot \mathbb{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 12 & 8 \\ 8 & 12 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} =$$
$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{T} \cdot \begin{bmatrix} 4 & 20 \\ -4 & 20 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 20 \\ -4 & 20 \end{bmatrix} =$$
$\frac{1}{2}\begin{bmatrix} 8 & 0 \\ 0 & 40 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix}$, a diagonal matrix.

I M 3) Given the linear function $F\colon \mathbb{R}^4 \to \mathbb{R}^3$, we know that for function F:

1. $F(1,0,0,0) = (1,0,0)$;
2. $F(1,1,0,0) = (1,1,0)$;
3. $F(1,1,1,0) = (0,0,0)$;
4. $F(1,1,1,1) = (0,0,0)$.

Calculate the matrix $\mathbb{A}_F$ associated at the linear function F and find the dimention of the kernel and the dimention of the image of F.

By 1., 2., 3. and 4. we can calculate the matrix $\underset{3\times 4}{\mathbb{A}_F}$ associated at function F,

$$\mathbb{A}_F = \begin{bmatrix} \alpha & \beta & \chi & \delta \\ \digamma & \epsilon & \eta & \gamma \\ \iota & \kappa & \mu & \nu \end{bmatrix}.$$

From 1. $F(1,0,0,0) = \begin{bmatrix} \alpha & \beta & \chi & \delta \\ \digamma & \epsilon & \eta & \gamma \\ \iota & \kappa & \mu & \nu \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \digamma \\ \iota \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$\alpha = 1$ and $\digamma = \iota = 0$; $\mathbb{A}_F = \begin{bmatrix} 1 & \beta & \chi & \delta \\ 0 & \epsilon & \eta & \gamma \\ 0 & \kappa & \mu & \nu \end{bmatrix}$.

From 2. $F(1,1,0,0) = \begin{bmatrix} 1 & \beta & \chi & \delta \\ 0 & \epsilon & \eta & \gamma \\ 0 & \kappa & \mu & \nu \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+\beta \\ \epsilon \\ \kappa \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow$

$\epsilon = 1$ and $\beta = \kappa = 0$; $\mathbb{A}_F = \begin{bmatrix} 1 & 0 & \chi & \delta \\ 0 & 1 & \eta & \gamma \\ 0 & 0 & \mu & \nu \end{bmatrix}$.

From 3. $F(1,1,1,0) = \begin{bmatrix} 1 & 0 & \chi & \delta \\ 0 & 1 & \eta & \gamma \\ 0 & 0 & \mu & \nu \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+\chi \\ 1+\eta \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$\chi = \eta = -1$ and $\mu = 0$; $\mathbb{A}_F = \begin{bmatrix} 1 & 0 & -1 & \delta \\ 0 & 1 & -1 & \gamma \\ 0 & 0 & 0 & \nu \end{bmatrix}$.

From 4. $F(1,1,1,1) = \begin{bmatrix} 1 & 0 & -1 & \delta \\ 0 & 1 & -1 & \gamma \\ 0 & 0 & 0 & \nu \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \delta \\ \gamma \\ \nu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$\delta = \gamma = \nu = 0$; $\mathbb{A}_F = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

To calculate the dimention of the image of linear function, we find the rank of matrix $\mathbb{A}_F$, trivially we can notice that matrix presents a null row, thus any 3 by 3 sub-matrix from $\mathbb{A}_F$ has determinant equal 0, while the 2 by 2 sub-matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ presents determinant different from 0, this implies that matrix $\mathbb{A}_F$ has rank 2, thus the dimention of the image of function F is 2 and the dimention of the kernel is $dim(Ker(F)) = dim(\mathbb{R}^4) - dim(Ima(F)) = 4 - 2 = 2$.

I M 4) Consider the matrix $\mathbb{U} = \begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}$. Knowing that the matrix $\mathbb{U}$ is a horthogonal matrix, calculate the value of α and find matrix $\mathbb{U}^{-1}$, the inverse matrix of $\mathbb{U}$.

Remember that a matrix $\mathbb{U}$ is a horthogonal matrix if and only if $\mathbb{U} \cdot \mathbb{U}^T = \mathbb{U}^T \cdot \mathbb{U} = \mathbb{I}$, an identity matrix. Also matrix $\mathbb{U}$ is symmetrical, so $\mathbb{U}^T = \mathbb{U}$ and $\mathbb{U} \cdot \mathbb{U}^T = \mathbb{U}^T \cdot \mathbb{U} = \mathbb{U} \cdot \mathbb{U}$.

$\mathbb{U} \cdot \mathbb{U} = \begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix} \cdot \begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix} = \begin{bmatrix} 1+\alpha^2 & 2\alpha \\ 2\alpha & 1+\alpha^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if and only if $\alpha = 0$;

$\mathbb{U} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. For the inverse matrix of $\mathbb{U}$, remember that matrix

$\mathbb{U}^{-1} = \frac{1}{|\mathbb{U}|} (Adj(\mathbb{U}))^T$, where $Adj(\mathbb{U})$ is the adjoin matrix of $\mathbb{U}$.

$\mathbb{U}^{-1} = -\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^T = \mathbb{U}$.

To check the result, we calculate the product $\mathbb{U}^{-1} \cdot \mathbb{U}$.

$\mathbb{U}^{-1} \cdot \mathbb{U} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, an identity matrix.

Note: by the condition $\mathbb{U} \cdot \mathbb{U}^T = \mathbb{U}^T \cdot \mathbb{U} = \mathbb{I}$ we get $\mathbb{U}^{-1} = \mathbb{U}^T = \mathbb{U}$, because $\mathbb{U}$ is symmetrical.

II M 1) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x + 4y \\ \text{u.c.: } x^2 + 4y^2 \leq 4 \end{cases}$.

The function f is an affine continuos function, the admissible region is an ellipse with center $(0,0)$, a bounded and closed set, therefore f presents absolute maximum and minimum in the admissible region. The Lagrangian function is

$\mathcal{L}(x,y,\lambda) = x + 4y - \lambda(x^2 + 4y^2 - 4)$ with

$\nabla\mathcal{L} = (1 - 2\lambda x, 4 - 8\lambda y, -(x^2 + 4y^2 - 4))$.

$I^\circ\ CASE$ ($free\ optimization$):

$\begin{cases} \lambda = 0 \\ 1 = 0 \\ 4 = 0 \\ x^2 + 4y^2 \leq 4 \end{cases}$. System impossible.

$II^\circ\ CASE$ ($constrained\ optimization$):

$$\begin{cases}\lambda \neq 0\\ 1-2\lambda x=0\\ 4-8\lambda y=0\\ x^2+4y^2=4\end{cases} \Rightarrow \begin{cases}\lambda \neq 0\\ x=\frac{1}{2\lambda}\\ y=\frac{1}{2\lambda}\\ \left(\frac{1}{2\lambda}\right)^2+4\left(\frac{1}{2\lambda}\right)^2=4\end{cases} \Rightarrow \begin{cases}\lambda \neq 0\\ x=\frac{1}{2\lambda}\\ y=\frac{1}{2\lambda}\\ \frac{5}{4\lambda^2}=4\end{cases} \Rightarrow \begin{cases}\lambda \neq 0\\ x=\frac{1}{2\lambda}\\ y=\frac{1}{2\lambda}\\ \lambda^2=\frac{5}{16}\end{cases} \Rightarrow$$

$\begin{cases}\lambda \neq 0\\ x=\pm\frac{2}{5}\sqrt{5}\\ y=\pm\frac{2}{5}\sqrt{5}\\ \lambda=\pm\frac{1}{4}\sqrt{5}\end{cases}$; two constrained critical points $P_{1,2}\left(\pm\frac{2}{5}\sqrt{5},\ \pm\frac{2}{5}\sqrt{5}\right)$. The first point presents $\lambda > 0$, point of maximum, the second presents $\lambda < 0$, point of mimimum. We get the maximum $f\left(\frac{2}{5}\sqrt{5}, \frac{2}{5}\sqrt{5}\right)=2\sqrt{5}$ and the minimum $f\left(-\frac{2}{5}\sqrt{5},\ -\frac{2}{5}\sqrt{5}\right)=-2\sqrt{5}$.

Alternative Solution: The function f is an affine continuos function, the admissible region is an ellipse with center $(0,0)$, a bounded, convex and closed set, this implies that the point of maximum and the point of minimum must be found on the border of the admissible region. We write the Lagrangian function of the problem as $\mathcal{L}(x,y,\lambda)=x+4y-\lambda(x^2+4y^2-4)$ with $\nabla\mathcal{L}=(1-2\lambda x, 4-8\lambda y, -(x^2+y^2-4))$.

FOC:

$$\begin{cases}1-2\lambda x=0\\ 4-8\lambda y=0\\ x^2+y^2=4\end{cases} \Rightarrow \begin{cases}x=\frac{1}{2\lambda}\\ y=\frac{1}{2\lambda}\\ \left(\frac{1}{2\lambda}\right)^2+4\left(\frac{1}{2\lambda}\right)^2=4\end{cases} \Rightarrow \begin{cases}x=\frac{1}{2\lambda}\\ y=\frac{1}{2\lambda}\\ \frac{5}{4\lambda^2}=4\end{cases} \Rightarrow \begin{cases}x=\pm\frac{2}{5}\sqrt{5}\\ y=\pm\frac{2}{5}\sqrt{5}\\ \lambda=\pm\frac{1}{4}\sqrt{5}\end{cases}$$

; two constraint critical points $P_{1,2}=\left(\pm\frac{2}{5}\sqrt{5},\ \pm\frac{2}{5}\sqrt{5}\right)$.

SOC:

$$\overline{\mathcal{H}}=\begin{bmatrix}0 & -2x & -8y\\ -2x & -2\lambda & 0\\ -8y & 0 & -8\lambda\end{bmatrix}, \text{ with } |\overline{\mathcal{H}}|=\begin{vmatrix}0 & -2x & -8y\\ -2x & -2\lambda & 0\\ -8y & 0 & -8\lambda\end{vmatrix}=$$

$$2x\cdot\begin{vmatrix}-2x & 0\\ -8y & -8\lambda\end{vmatrix}-8y\cdot\begin{vmatrix}-2x & -2\lambda\\ -8y & 0\end{vmatrix}=32\lambda x^2+128\lambda y^2=32\lambda(x^2+4y^2).$$

$|\overline{\mathcal{H}}(P_1)|=32\sqrt{5}>0$, P_1 point of maximum with $f(P_1)=2\sqrt{5}$,

$|\overline{\mathcal{H}}(P_2)|=-32\sqrt{5}<0$, P_2 point of minimum with $f(P_2)=-2\sqrt{5}$.

II M 2) Given the system of equations $\begin{cases}x^2+y^2+z^2=3\\ x^3+2y^3+3z^3=0\end{cases}$, satisfied at the point $P(1,1,-1)$; verify that with it an implicit function $z\mapsto(x(z),y(z))$ can be defined and then calculate, for this implicit function, the derivatives $x'(-1)$ and $y'(-1)$.

On point P $\begin{cases}1^2+1^2+(-1)^2=3\\ 1^3+2\cdot 1^3+3\cdot(-1)^3=0\end{cases}$, system is satisfied. Consider now the function $F:\mathbb{R}^3\to\mathbb{R}^2$ with $F(x,y,z)=(x^2+y^2+z^2, x^3+2y^3+3z^3)$ and the Jacobian of F equal $\mathcal{J}F=\begin{bmatrix}2x & 2y & 2z\\ 3x^2 & 6y^2 & 9z^2\end{bmatrix}$ with $\mathcal{J}F(P)=\begin{bmatrix}2 & 2 & -2\\ 3 & 6 & 9\end{bmatrix}$, and restricted $\mathcal{J}F(P)$ in relation to the variables (x,y): $\mathcal{J}F(P)\Big|_{(x,y)}=\begin{bmatrix}2 & 2\\ 3 & 6\end{bmatrix}$ with determinant equal $6 \neq 0$. The proposed system defines in a neighbourhood of point P an implicit function $z \mapsto (x(z), y(z))$; for the two derivatives we have:

$$\begin{pmatrix} x'(-1) \\ y'(-1) \end{pmatrix} = -\left(\mathcal{J}F(P)\Big|_{(x,y)}\right)^{-1} \cdot \mathcal{J}F(P)\Big|_{z} = -\begin{bmatrix} 2 & 2 \\ 3 & 6 \end{bmatrix}^{-1} \cdot \begin{pmatrix} -2 \\ 9 \end{pmatrix} =$$

$$-\frac{1}{6}\begin{bmatrix} 6 & -3 \\ -2 & 2 \end{bmatrix}^{T} \cdot \begin{pmatrix} -2 \\ 9 \end{pmatrix} = -\frac{1}{6}\begin{bmatrix} 6 & -2 \\ -3 & 2 \end{bmatrix} \cdot \begin{pmatrix} -2 \\ 9 \end{pmatrix} =$$

$$-\frac{1}{6}\begin{pmatrix} -30 \\ 24 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}.$$

Alternative Method: the two derivatives can be achieved by the system

$\begin{cases} 2\,\frac{dx}{dz} + 2\,\frac{dy}{dz} - 2\,\frac{dz}{dz} = 0 \\ 3\,\frac{dx}{dz} + 6\,\frac{dy}{dz} + 9\,\frac{dz}{dz} = 0 \end{cases}$, that calculated in the point P can be written as

$\begin{cases} 2\,x'(-1) + 2\,y'(-1) - 2 = 0 \\ 3\,x'(-1) + 6\,y'(-1) + 9 = 0 \end{cases}$. Appling Cramer's Rule we get:

$$x'(-1) = -\frac{\begin{vmatrix} -2 & 2 \\ 9 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 3 & 6 \end{vmatrix}} = -\frac{-30}{6} = 5 \text{ and}$$

$$y'(-1) = -\frac{\begin{vmatrix} 2 & -2 \\ 3 & 9 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 3 & 6 \end{vmatrix}} = -\frac{24}{6} = -4\,.$$

II M 3) Given the function $f(x, y) = ax^2 + 2xy + by^2$, we know that the directional derivatives $\mathcal{D}_v f(1, 1)$ and $\mathcal{D}_w f(1, 1)$ are both equal 0, where v is the unit vector $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and w is the unit vector $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. Find the values of the parametes a and b.

Function f is differentiable at any point (x, y) with gradient vector $\nabla f(x, y) = (2ax + 2y, 2x + 2by)$ and $\nabla f(1, 1) = (2a + 2, 2 + 2b)$, the two directional derivatives are $\mathcal{D}_v f(1, 1) = (2a + 2, 2 + 2b) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \sqrt{2}\,(a + b + 2)$ and $\mathcal{D}_w f(1, 1)(2a + 2, 2 + 2b) \cdot \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \sqrt{2}\,(a - b)$; put $\sqrt{2}\,(a + b + 2) = 0$ and $\sqrt{2}\,(a - b) = 0$ easily we find $a = b = -1$.

II M 4) Calculate the partial derivatives of function $f(x, y, z, w) = e^{zw^3} - 3xy$.

$$f'_x = -3y \qquad f'_y = -3x \qquad f'_z = w^3 e^{zw^3} \qquad f'_w = 3zw^2 e^{zw^3}\,.$$