UNIVERSITA' DEGLI STUDI DI SIENA
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Intermediate Test Quantitative Methods for Economic Applications - Mathematics (07/11/24)

1) Given the matrix: $\mathbb{A} = \begin{bmatrix} 2 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$; calculate its eigenvalues and study if the matrix is diagonalizable.

We start with the calculation of characteristic polynomial of matrix $\mathbb{A}$;

$P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda-2 & -2 & 0 & 0 \\ 2 & \lambda-2 & 0 & 0 \\ 0 & 0 & \lambda-1 & -1 \\ 0 & 0 & 1 & \lambda-1 \end{vmatrix} =$

$\begin{vmatrix} \lambda-2 & -2 \\ 2 & \lambda-2 \end{vmatrix} \cdot \begin{vmatrix} \lambda-1 & -1 \\ 1 & \lambda-1 \end{vmatrix} = \left((\lambda-2)^2+4\right)\left((\lambda-1)^2+1\right)$. Putting $(\lambda-2)^2+4=0$ we get $\lambda-2=\pm 2i$ and the first two eigenvalues of matrix $\mathbb{A}$ are $\lambda_{1,2}=2\pm 2i$, while putting $(\lambda-1)^2+1=0$ we get $\lambda-1=\pm i$ and the second two eigenvalues of matrix $\mathbb{A}$ are $\lambda_{3,4}=1\pm i$. Matrix $\mathbb{A}$ has four eigenvalues with any of one different from the others, thus matrix $\mathbb{A}$ is a diagonalizable one.

2) Calculate the two square roots $\sqrt{i^{13}-i^{15}}$.

$\sqrt{i^{13}-i^{15}} = \sqrt{i^{13}(1-i^2)} = \sqrt{i\cdot i^{12}\cdot 2} = \sqrt{2i} = \sqrt{2\left(cos\frac{\pi}{2}+i\,sin\frac{\pi}{2}\right)}$. Appling De Moivre Formula we get: $z_k = \sqrt{2}\left(cos\left(\frac{\pi/2+2k\pi}{2}\right)+i\,sin\left(\frac{\pi/2+2k\pi}{2}\right)\right) =$ $\sqrt{2}\left(cos\left(\frac{\pi}{4}+k\pi\right)+i\,sin\left(\frac{\pi}{4}+k\pi\right)\right)$ $k=0,1$. The two roots are:

$k=0 \rightarrow z_0 = \sqrt{2}\left(cos\frac{\pi}{4}+i\,sin\frac{\pi}{4}\right) = 1+i$;

$k=1 \rightarrow z_1 = \sqrt{2}\left(cos\frac{5}{4}\pi+i\,sin\frac{5}{4}\pi\right) = -(1+i) = -z_0$.

3) Given a linear map $F\colon \mathbb{R}^4 \rightarrow \mathbb{R}^3$, with $F(x_1,x_2,x_3,x_4) = (kx_1+x_3-x_4, -2x_2+x_3-x_4, 2x_1+2kx_2-x_3+x_4)$; where k is a real parameter. Study, varying k, if the linear map is surjective and in the case $k=2$, find a basis for the image and a basis for the kernel of F.

The matrix associated with the linear map F is $\mathbb{A}_F = \begin{bmatrix} k & 0 & 1 & -1 \\ 0 & -2 & 1 & -1 \\ 2 & 2k & -1 & 1 \end{bmatrix}$; we reduce the matrix by elementary operations on its lines: $\begin{bmatrix} k & 0 & 1 & -1 \\ 0 & -2 & 1 & -1 \\ 2 & 2k & -1 & 1 \end{bmatrix} C_1 \leftrightarrows C_3$

$$\begin{bmatrix} 1 & 0 & k & -1 \\ 1 & -2 & 0 & -1 \\ -1 & 2k & 2 & 1 \end{bmatrix} \begin{matrix} C_3 \to C_3 - k\cdot C_1 \\ C_4 \to C_4 + C_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & -k & 0 \\ -1 & 2k & 2+k & 0 \end{bmatrix} C_3 \to C_3 - \tfrac{k}{2}\cdot C_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 2k & 2+k-k^2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 2k & (1+k)(2-k) & 0 \end{bmatrix}.$$ As we can observe by the last matrix, $Rank(\mathbb{A}_F) = Dim(Ima(F)) = \begin{cases} 3 & \text{if } k \neq -1 \wedge k \neq 2 \\ 2 & \text{otherwise} \end{cases}$,
thus the linear map is surjective if and only if $k \neq -1 \wedge k \neq 2$. In the case $k = 2$, $Dim(Ima(F)) = 2$ and $Dim(Ker(F)) = Dim(\mathbb{R}^4) - 2 = 2$; for a basis of the kernel observe that a generic element in the kernel must satisfied the system

$$\begin{cases} 2x_1 + x_3 - x_4 = 0 \\ -2x_2 + x_3 - x_4 = 0 \\ 2x_1 + 4x_2 - x_3 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_4 = 2x_1 + x_3 \\ -2x_2 + x_3 - (2x_1 + x_3) = 0 \\ 2x_1 + 4x_2 - x_3 + 2x_1 + x_3 = 0 \end{cases} \Rightarrow$$

$$\begin{cases} x_4 = 2x_1 + x_3 \\ x_1 + x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_4 = 2x_1 + x_3 \\ x_2 = -x_1 \end{cases}$$; a generic element in the kernel is $(x_1, -x_1, x_3, 2x_1 + x_3) = x_1(1, -1, 0, 2) + x_3(0, 0, 1, 1)$, a basis for the kernel is the set $\mathcal{B}_{Ker(F)} = \{(1, -1, 0, 2), (0, 0, 1, 1)\}$. Consider now a generic element in the image (y_1, y_2, y_3), its must satisfied the system $\begin{cases} 2x_1 + x_3 - x_4 = y_1 \\ -2x_2 + x_3 - x_4 = y_2 \\ 2x_1 + 4x_2 - x_3 + x_4 = y_3 \end{cases}$; it is not difficult observe that $y_1 - 2y_2 = 2x_1 + x_3 - x_4 - 2(-2x_2 + x_3 - x_4) = 2x_1 + 4x_2 - x_3 + x_4 = y_3$, easily we conclude that a generic element in the image is $(y_1, y_2, y_3) = (y_1, y_2, y_1 - 2y_2) = y_1(1, 0, 1) + y_2(0, 1, -2)$; a basis for the image is the set $\mathcal{B}_{Ima(F)} = \{(1, 0, 1), (0, 1, -2)\}$.

4) Given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ belong on $\mathbb{R}^n$ and consider the map $F\colon \mathbb{R}^n \to \mathbb{R}$ with $\forall \boldsymbol{x} \in \mathbb{R}^n, F(\boldsymbol{x}) = \ <\boldsymbol{a}, \boldsymbol{x}> - <\boldsymbol{b}, \boldsymbol{x}>$. Where with $<\boldsymbol{y}, \boldsymbol{z}>$ we indicate the scalar product between the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$. Prove that F is a linear map.

By the properties of the scalar product, $\forall(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\forall(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} F(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) &= \ <\boldsymbol{a}, \alpha\boldsymbol{x} + \beta\boldsymbol{y}> - <\boldsymbol{b}, \alpha\boldsymbol{x} + \beta\boldsymbol{y}> \\ &= \ <\boldsymbol{a}, \alpha\boldsymbol{x}> + <\boldsymbol{a}, \beta\boldsymbol{y}> - (<\boldsymbol{b}, \alpha\boldsymbol{x}> + <\boldsymbol{b}, \beta\boldsymbol{y}>) \\ &= \alpha <\boldsymbol{a}, \boldsymbol{x}> + \beta <\boldsymbol{a}, \boldsymbol{y}> - (\alpha <\boldsymbol{b}, \boldsymbol{x}> + \beta <\boldsymbol{b}, \boldsymbol{y}>) \\ &= \alpha(<\boldsymbol{a}, \boldsymbol{x}> - <\boldsymbol{b}, \boldsymbol{x}>) + \beta(<\boldsymbol{a}, \boldsymbol{y}> - <\boldsymbol{b}, \boldsymbol{y}>) \\ &= \alpha F(\boldsymbol{x}) + \beta F(\boldsymbol{y}). \end{aligned}$$ F is a linear map.