UNIVERSITÁ DEGLI STUDI DI SIENA
Scuola di Economia e Management
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Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 10/1/2025

I M 1) Given the complex number $z = -8 - 8\sqrt{3}\,i$, calculate $\sqrt[4]{z}$.

We start with the calculus of ρ_z, the modulus of z: $\rho_z = \sqrt{(-8)^2 + \left(-8\sqrt{3}\right)^2} = \sqrt{64+192} = \sqrt{256} = 16$; for the argument of z we note that z belongs to the third quarter of the complex plane, and we calculate its by the inverse tangent function:

$\theta_z = \pi + tan^{-1}\left(\dfrac{-8\sqrt{3}}{-8}\right) = \pi + tan^{-1}\left(\sqrt{3}\right) = \pi + \dfrac{\pi}{3} = \dfrac{4\pi}{3}$; it follows

$z = 16\left(cos\left(\dfrac{4\pi}{3}\right) + i\,sin\left(\dfrac{4\pi}{3}\right)\right)$. For the roots we apply the classical formula:

$$\sqrt[4]{z} = z_k = \sqrt[4]{16}\left(cos\left(\frac{4\pi/3 + 2k\pi}{4}\right) + i\,sin\left(\frac{4\pi/3 + 2k\pi}{4}\right)\right) \;\; k = 0,1,2,3$$
$$= 2\left(cos\left(\frac{\pi}{3} + k\frac{\pi}{2}\right) + i\,sin\left(\frac{\pi}{3} + k\frac{\pi}{2}\right)\right) \;\; k = 0,1,2,3.$$

The four roots are:

$k = 0 \rightarrow\; z_0 = 2\left(cos\dfrac{\pi}{3} + i\,sin\dfrac{\pi}{3}\right) = 2\left(\dfrac{1}{2} + \dfrac{1}{2}\sqrt{3}i\right) = 1 + \sqrt{3}i$;

$k = 1 \rightarrow\; z_1 = 2\left(cos\dfrac{5\pi}{6} + i\,sin\dfrac{5\pi}{6}\right) = 2\left(-\dfrac{1}{2}\sqrt{3} + \dfrac{1}{2}i\right) = -\sqrt{3} + i$;

$k = 2 \rightarrow\; z_2 = 2\left(cos\dfrac{4\pi}{3} + i\,sin\dfrac{4\pi}{3}\right) = 2\left(-\dfrac{1}{2} - \dfrac{1}{2}\sqrt{3}i\right) = -\left(1 + \sqrt{3}i\right)$;

$k = 3 \rightarrow\; z_3 = 2\left(cos\dfrac{11\pi}{6} + i\,sin\dfrac{11\pi}{6}\right) = 2\left(\dfrac{1}{2}\sqrt{3} - \dfrac{1}{2}i\right) = \sqrt{3} - i$.

I M 2) Given the matrix $\mathbb{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & k \\ 4 & -4 & 5 \end{bmatrix}$. Calculate the value of k such that matrix $\mathbb{A}$ has determinant equal 0, and then study if the matrix $\mathbb{A}$ is diagonalizable or not.

The determinant of matrix $\mathbb{A}$ is $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & k \\ 4 & -4 & 5 \end{vmatrix} = -2\begin{vmatrix} 2 & -1 \\ -4 & 5 \end{vmatrix} - k\begin{vmatrix} 1 & 2 \\ 4 & -4 \end{vmatrix} =$ $-2(10-4) - k(-4-8) = -12 + 12k$, put $-12 + 12k = 0$ easily we find $k = 1$,

$\mathbb{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$. To study the diagonalizability of $\mathbb{A}$ we start with the calculus of the characteristic polynomial of the matrix;

$$P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda - 1 & -2 & 1 \\ -2 & \lambda & -1 \\ -4 & 4 & \lambda - 5 \end{vmatrix} =$$

$$(\lambda - 1)\begin{vmatrix} \lambda & -1 \\ 4 & \lambda - 5 \end{vmatrix} + 2\begin{vmatrix} -2 & -1 \\ -4 & \lambda - 5 \end{vmatrix} + \begin{vmatrix} -2 & \lambda \\ -4 & 4 \end{vmatrix} =$$

$(\lambda - 1)(\lambda(\lambda - 5) + 4) + 2(-2(\lambda - 5) - 4) - 8 + 4\lambda =$
$(\lambda - 1)(\lambda^2 - 5\lambda + 4) + 2(-2\lambda + 6) - 8 + 4\lambda =$
$\lambda^3 - 5\lambda^2 + 4\lambda - \lambda^2 + 5\lambda - 4 - 4\lambda + 12 - 8 + 4\lambda = \lambda^3 - 6\lambda^2 + 9\lambda =$
$\lambda(\lambda^2 - 6\lambda + 9) = \lambda(\lambda - 3)^2$. Putting $P_{\mathbb{A}}(\lambda) = 0$ we find the three eigenvalues of matrix $\mathbb{A}$: $\lambda_1 = 0$, $\lambda_{2,3} = 3$, the eigenvalue 3 has algebraic multiplicity equal two. To verify if the matrix is diagonalizable, we must find the geometric multiplicity of eigenvalue 3, for our goal we calculate the rank of matrix

$3\mathbb{I} - \mathbb{A} = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 3 & -1 \\ -4 & 4 & -2 \end{bmatrix}$, it's easy note that from matrix $\mathbb{A}$ we can define a principal minor of order 2, $\begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}$ with determinant different from 0 and matrix $3\mathbb{I} - \mathbb{A}$ has the third raw equal to the opposite of the double of the first raw, thus $Rank(\mathbb{A}) = 2$ and the geometric multiplicity of eigenvalue 3 is one. The matrix isn't diagonalizable.

I M 3) Given a linear map $F: \mathbb{R}^4 \to \mathbb{R}^3$, with $F(X) = A \cdot X$ and $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & k \\ 1 & 1 & 1 & m \end{bmatrix}$. Determine the values of the parameters k and m such that the dimension of the Kernel is maximum, and then find a basis for the Kernel and a basis for the Image of such linear map.

From the Rank-Nullity Theorem $dim(Ima(F)) + dim(Ker(F)) = dim(\mathbb{R}^4) = 4$, so the dimension of the Kernel is maximum if and only if the dimension of the Image is smallest, and the dimension of the Image is equal to the rank of the associated matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & k \\ 1 & 1 & 1 & m \end{bmatrix}$. We reduce A by elementary operations on its lines:

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & k \\ 1 & 1 & 1 & m \end{bmatrix} \begin{matrix} R_2 \mapsto R_2 - R_1 \\ R_3 \mapsto R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & k - 1 \\ 0 & 0 & 0 & m - 1 \end{bmatrix}$, from the reduced matrix we find that $Rank(A) = \begin{cases} 1 & \text{if } k = m = 1 \\ 2 & \text{otherwise} \end{cases}$ and so the dimension of the Kernel is biggest if $k = m = 1$.

To find a basis for the kernel we know that a vector (x, y, z, w) belongs to the kernel of linear map F if and only if $F(x, y, z, w) = (0, 0, 0)$, but

$$F(x, y, z, w) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x + y + z + w \\ x + y + z + w \\ x + y + z + w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$w = -x - y - z$. Thus a generic vector that belongs to the kernel is a vector $(x, y, z, -x - y - z) = x(1, 0, 0, -1) + y(0, 1, 0, -1) + z(0, 0, 1, -1)$, and a basis for the kernel is the set $\mathcal{B}_{Ker(F)} = \{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1)\}$. While for the basis of the image of F, we have that a generic element of the image is $(b,b,b)=b(1,1,1)$, and a basis for the image is the set $\mathcal{B}_{Ima(F)}=\{(1,1,1)\}$.

I M 4) Matrix $\mathbb{A}=\begin{bmatrix}1&0&x\\0&y&0\\z&0&1\end{bmatrix}$ has inverse matrix $\mathbb{A}^{-1}=\begin{bmatrix}1/3&0&2/3\\0&1&0\\-1/3&0&1/3\end{bmatrix}$. Find the values of the three parameters x, y and z.

Remember that if $\mathbb{A}^{-1}$ is the inverse matrix of matrix $\mathbb{A}$; $\mathbb{A}\cdot\mathbb{A}^{-1}=\mathbb{A}^{-1}\cdot\mathbb{A}=\mathbb{I}_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$. We calculate the product $\mathbb{A}\cdot\mathbb{A}^{-1}$; $\begin{bmatrix}1&0&x\\0&y&0\\z&0&1\end{bmatrix}\cdot\begin{bmatrix}1/3&0&2/3\\0&1&0\\-1/3&0&1/3\end{bmatrix}=$

$\begin{bmatrix}\frac{1-x}{3}&0&\frac{2+x}{3}\\0&y&0\\\frac{z-1}{3}&0&\frac{2z+1}{3}\end{bmatrix}$ and $\begin{bmatrix}\frac{1-x}{3}&0&\frac{2+x}{3}\\0&y&0\\\frac{z-1}{3}&0&\frac{2z+1}{3}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ if $\begin{cases}\frac{1-x}{3}=1\\\frac{2+x}{3}=0\\y=1\\\frac{z-1}{3}=0\\\frac{2z+1}{3}=1\end{cases}$; we conclude that the three parameters are $x=-2$ and $y=z=1$; $\mathbb{A}=\begin{bmatrix}1&0&-2\\0&1&0\\1&0&1\end{bmatrix}$. To verify the result we calculate the product $\mathbb{A}^{-1}\cdot\mathbb{A}=\begin{bmatrix}1/3&0&2/3\\0&1&0\\-1/3&0&1/3\end{bmatrix}\cdot\begin{bmatrix}1&0&-2\\0&1&0\\1&0&1\end{bmatrix}=$

$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$.

II M 1) The equation $f(x,y)=x^3-y^3=0$ defines at point $P\left(-\frac{1}{3},-\frac{1}{3}\right)$ an implicity function $y=y(x)$. Calculate $y'\left(-\frac{1}{3}\right)$ and $y''\left(-\frac{1}{3}\right)$.

$f(P)=\left(-\frac{1}{3}\right)^3-\left(-\frac{1}{3}\right)^3=-\frac{1}{27}+\frac{1}{27}=0$, $f'_x=3x^2$ and $f'_y=-3y^2$, with $f'_x(P)=\frac{1}{3}$ and $f'_y(P)=-\frac{1}{3}$. Since $f'_y(P)\neq 0$, the equation $f(x,y)=x^3-y^3=0$ defines a function $y=y(x)$ with $y'\left(-\frac{1}{3}\right)=-\frac{f'_x(P)}{f'_y(P)}=1$. For the second order derivative we have

$$y''\left(-\frac{1}{3}\right)=-\frac{f''_{x,x}(P)+2\cdot f''_{x,y}(P)\cdot y'\left(-\frac{1}{3}\right)+f''_{y,y}(P)\cdot\left(y'\left(-\frac{1}{3}\right)\right)^2}{f'_y(P)}=$$

$3\left(f''_{x,x}(P)+2\cdot f''_{x,y}(P)+f''_{y,y}(P)\right)$ because $y'\left(-\frac{1}{3}\right)=1$ and $f'_y(P)=-\frac{1}{3}$. The second order partial derivatives are: $f''_{x,x}=6x$, $f''_{x,y}=0$ and $f''_{y,y}=-6y$ with $f''_{x,x}(P)=-2$ and $f''_{y,y}(P)=2$; $y''\left(-\frac{1}{3}\right)=3(-2+2\cdot 0+2)=0$.

II M 2) Solve the problem $\begin{cases}\text{Max/min } f(x,y)=x^2-2y^2\\\text{u.c.: } x^2+y^2\le 4\end{cases}$.

The function f is a polynomial, continuos function, the admissible region is a disk with center $(0,0)$ and radius 2, a bounded and closed set, therefore f presents absolute

maximum and minimum in the admissible region, constraint is qualified on any point in the circumference $x^2+y^2=4$. The Lagrangian function is
$\mathcal{L}(x,y,\lambda)=x^2-2\,y^2-\lambda(x^2+y^2-4)$ with
$\nabla\mathcal{L}=(2x-2\lambda x,\,-4y-2\lambda y,\,-(x^2+y^2-4))$.
$I^\circ\ CASE$ $(free\ optimization)$:

$$\begin{cases}\lambda=0\\2x=0\\-4y=0\\x^2+y^2\le 4\end{cases}\Rightarrow\begin{cases}\lambda=0\\x=0\\y=0\\0^2+0^2\le 4\end{cases}$$; point $(0,0)$ is admissible, $\mathcal{H}f=\begin{bmatrix}2&0\\0&-4\end{bmatrix}$,

$\mathcal{H}_2=-8<0$. $(0,0)$ is a saddle point.
$II^\circ\ CASE$ $(constrained\ optimization)$:

$$\begin{cases}\lambda\ne 0\\2x-2\lambda x=0\\-4y-2\lambda y=0\\x^2+y^2=4\end{cases}\Rightarrow\begin{cases}\lambda\ne 0\\2x(1-\lambda)=0\\-2y(2+\lambda)=0\\x^2+y^2=4\end{cases}$$; if $x=0$, $y=\pm 2$ and $\lambda=-2$,

otherwise if $y=0$, $x=\pm 2$ and $\lambda=1$. Four critical points $P_{1,2}=(0,\pm 2)$, both candidate for minimum ($\lambda<0$), and $P_{3,4}=(\pm 2,0)$, both candidate for maximum ($\lambda>0$). $f(P_{1,2})=-8$, $f(P_{3,4})=4$, f presents absolute maximum equal 4 on points $(\pm 2,0)$ and absolute minimum equal -8 on points $(0,\pm 2)$.

II M 3) Find the nature of the unique critical point of the function
$f(x,\,y)=x^2-y^2-xy^2$.
$\nabla f=\left(2x-y^2,\,-2y-2xy\right)$.
FOC:

$$\begin{cases}2x-y^2=0\\-2y-2xy=0\end{cases}\Rightarrow\begin{cases}2x=y^2\\-2y(1+x)=0\end{cases}$$; if $y=0$ then $x=0$, otherwise if $x=-1$ then $y^2=-2$, impossible. One critical point $O=(0,0)$.

SOC:

$\mathcal{H}_f=\begin{bmatrix}2&-2y\\-2y&-2-2x\end{bmatrix}$, with $|\mathcal{H}_f|=\begin{vmatrix}2&-2y\\-2y&-2-2x\end{vmatrix}=-4-4x-4y^2=-4(1+x+y^2)$; $|\mathcal{H}_f(O)|=-4<0$, O is a saddle point for function $f(x,\,y)$.

II M 4) Given the function $f(x,y)=xye^{x-y}$ and the unit vector $v=(sin\,\alpha,cos\,\alpha)$; knowing that at point (1, 1) the directional derivative $\mathcal{D}_v f(1,1)=2$, find the value of α and compute the second order directional derivative $\mathcal{D}^{(2)}_{v,v}f(1,1)$.
$\nabla f(x,y)=(ye^{x-y}+xye^{x-y},xe^{x-y}-xye^{x-y})=(y(1+x)e^{x-y},x(1-y)e^{x-y})$,
$\nabla f(1,1)=(1\cdot(1+1)\cdot e^0,1\cdot(1-1)\cdot e^0)=(2,0)$;
$\mathcal{D}_v f(1,1)=\nabla f(1,1)\cdot v=(2,0)\cdot(sin\,\alpha,cos\,\alpha)=2\,sin\,\alpha$. Put $2\,sin\,\alpha=2$ we get $sin\,\alpha=1$ and $\alpha=\dfrac{\pi}{2}$, $v=\left(sin\,\dfrac{\pi}{2},cos\,\dfrac{\pi}{2}\right)=(1,0)$.
$\mathcal{D}^{(2)}_{v,v}f(x,y)=v^T\cdot\mathcal{H}f(x,y)\cdot v=(1,0)\cdot\begin{bmatrix}f''_{x,x}&f''_{x,y}\\f''_{y,x}&f''_{y,y}\end{bmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}=f''_{x,x}(x,y)$ with
$\mathcal{D}^{(2)}_{v,v}f(1,1)=f''_{x,x}(1,1)$; now we calculate the second pure partial derivative respect x:
$f''_{x,x}=ye^{x-y}+y(1+x)e^{x-y}=y(2+x)e^{x-y}$,
$\mathcal{D}^{(2)}_{v,v}f(1,1)=f''_{x,x}(1,1)=1\cdot(2+1)\cdot e^0=3$.