UNIVERSITÁ DEGLI STUDI DI SIENA
Scuola di Economia e Management
A.A. 2024/25
Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 4/2/2025

I M 1) Given the complex numbers $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$ and $w = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$, calculate the square roots of the complex number $y = z^{12} \cdot w^{10}$.

The complex numbers $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$ and $w = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$ have modulus respectively $\rho_z = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1$ and $\rho_w = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1$, for the arguments we observe that z belongs to the first quarter of the immaginary plane and $\theta_z = tan^{-1}\left(\frac{\sqrt{2}/2}{\sqrt{2}/2}\right) = \frac{\pi}{4}$, while w belongs to the second quarter of the immaginary plane and $\theta_w = \pi + tan^{-1}\left(\frac{\sqrt{2}/2}{-\sqrt{2}/2}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$; thus in exponential form, $z = 1 \cdot e^{\frac{\pi}{4} i}$ and $w = 1 \cdot e^{\frac{3\pi}{4} i}$. For the complex number y we get $y = z^{12} \cdot w^{10} = \left(1 \cdot e^{\frac{\pi}{4} i}\right)^{12} \cdot \left(1 \cdot e^{\frac{3\pi}{4} i}\right)^{10} = e^{3\pi i} \cdot e^{\frac{15\pi}{2} i} = e^{\frac{21\pi}{2} i} = cos\left(\frac{21\pi}{2}\right) + i\, sin\left(\frac{21\pi}{2}\right) = cos\left(\frac{\pi}{2}\right) + i\, sin\left(\frac{\pi}{2}\right)$. For the roots we apply the classical formula:

$$\sqrt{y} = y_k = cos\left(\frac{\pi/2 + 2k\pi}{2}\right) + i\, sin\left(\frac{\pi/2 + 2k\pi}{2}\right) \;\; k = 0, 1$$
$$= cos\left(\frac{\pi}{4} + k\pi\right) + i\, sin\left(\frac{\pi}{4} + k\pi\right) \;\; k = 0, 1.$$

The two roots are:

$k = 0 \rightarrow \; y_0 = cos\left(\frac{\pi}{4}\right) + i\, sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i = z$;

$k = 1 \rightarrow \; y_1 = cos\left(\frac{5\pi}{4}\right) + i\, sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i = -z$.

Alternative solution: $z \cdot w = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) \cdot \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = \left(\frac{\sqrt{2}}{2} i\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = -\frac{1}{2} - \frac{1}{2} = -1$; thus $y = z^{12} \cdot w^{10} = z^2 (z \cdot w)^{10} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)^2 \cdot (-1)^{10} = \frac{1}{2} + i - \frac{1}{2} = i = cos\left(\frac{\pi}{2}\right) + i\, sin\left(\frac{\pi}{2}\right)$. The calculus of the two roots follows as the previous page.

I M 2) Given the matrix $\mathbb{A}=\begin{bmatrix}2&2&1\\0&1&0\\1&2&2\end{bmatrix}$. Study if the matrix is diagonalizable and for eigenvalue $\lambda=1$ calculate a basis for the associated eigenspace.
To study the diagonalizability of $\mathbb{A}$ we start with the calculus of the characteristic polynomial of the matrix; $P_{\mathbb{A}}(\lambda)=|\lambda\mathbb{I}-\mathbb{A}|=\begin{vmatrix}\lambda-2&-2&-1\\0&\lambda-1&0\\-1&-2&\lambda-2\end{vmatrix}=$
$(\lambda-1)\begin{vmatrix}\lambda-2&-1\\-1&\lambda-2\end{vmatrix}=(\lambda-1)\big((\lambda-2)^2-1\big)=(\lambda-1)(\lambda^2-4\lambda+3)=$
$(\lambda-1)(\lambda-1)(\lambda-3)=(\lambda-1)^2(\lambda-3)$. Putting $P_{\mathbb{A}}(\lambda)=0$ we find the three eigenvalues of matrix $\mathbb{A}$: $\lambda_{1,2}=1$, $\lambda_3=3$, the eigenvalue 1 has algebraic multiplicity equal two. To verify if the matrix is diagonalizable, we must find the geometric multiplicity of eigenvalue 1, for our goal we calculate the rank of matrix
$1\cdot\mathbb{I}-\mathbb{A}=\begin{bmatrix}-1&-2&-1\\0&0&0\\-1&-2&-1\end{bmatrix}$, it's easy to note that matrix $1\cdot\mathbb{I}-\mathbb{A}$ has a null row and two rows identical, thus $Rank(1\cdot\mathbb{I}-\mathbb{A})=1$ and the geometric multiplicity of eigenvalue 1 is two. The matrix is diagonalizable.
To calculate a basis for the associated eigenspace of eigenvalue $\lambda=1$ we consider the equality $\begin{bmatrix}-1&-2&-1\\0&0&0\\-1&-2&-1\end{bmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ that implies $-x-2y-z=0$ or $z=-x-2y$, a generic vector that belongs to the associated eigenspace of eigenvalue $\lambda=1$ is $(x,y,-x-2y)=x(1,0,-1)+y(0,1,-2)$; a basis for the associated eigenspace is the set of vectors $\mathcal{B}_{\lambda=1}=\{(1,0,-1),(0,1,-2)\}$.
I M 3) Given a linear map $F\colon\mathbb{R}^4\to\mathbb{R}^4$, with $F(X)=A\cdot X$ and
$A=\begin{bmatrix}1&0&0&k\\0&1&k&0\\0&m&1&0\\m&0&0&1\end{bmatrix}$. Determine the values of the parameters k and m knowing that the vector $(1,-1,-1,1)$ belongs to the Kernel of F and find the dimension of the Image and the dimension of the Kernel of F.
If vector $(1,-1,-1,1)$ belongs to the Kernel of F, $F(1,-1,-1,1)=\mathbb{O}$ with
$F(1,-1,-1,1)=\begin{bmatrix}1&0&0&k\\0&1&k&0\\0&m&1&0\\m&0&0&1\end{bmatrix}\cdot\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}=\begin{pmatrix}1+k\\-1-k\\-m-1\\m+1\end{pmatrix}$; put
$\begin{pmatrix}1+k\\-1-k\\-m-1\\m+1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ easily we find $k=m=-1$ and matrix
$A=\begin{bmatrix}1&0&0&-1\\0&1&-1&0\\0&-1&1&0\\-1&0&0&1\end{bmatrix}$. To find the dimension of the Image and the dimension of the Kernel of F we calculate, by elementary operations on the lines, the

rank of matrix A: $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_3 \mapsto R_3 + R_2 \\ R_4 \mapsto R_4 + R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; the reduced matrix has two null rows and a two by two sub-matrix with determinant equal one, thus $Rank(A) = 2$ and from the Rank-Nullity Theorem we conclude that the dimension of the Image and the dimension of the Kernel of F are both equal two.

I M 4) Vector $V \in \mathbb{R}^3$ has coordinates $(3, -2, 1)$ respect the basis $\mathcal{B} = \{(1,1,1), (0,1,0), (0,0,1)\}$. Find the coordinates of vector V respect the new basis $\mathcal{B}' = \{(1,0,0), (0,1,0), (-1, -1, -1)\}$.

If vector V has coordinates $(3, -2, 1)$ respect the basis $\mathcal{B}$, $V = 3(1,1,1) - 2(0,1,0) + 1(0,0,1) = (3,1,4)$; if (α, β, χ) are the coordinates of V respect the basis $\mathcal{B}'$, $V = \alpha(1,0,0) + \beta(0,1,0) + \chi(-1, -1, -1) = (\alpha - \chi, \beta - \chi, -\chi)$. Putting $(\alpha - \chi, \beta - \chi, -\chi) = (3,1,4)$ we get $\alpha = -1$, $\beta = -3$, and $\chi = -4$.

Alternative solution: if $(3, -2, 1)$ are the coordinates of vector V respect the basis $\mathcal{B}$ and (α, β, χ) are those respect the basis $\mathcal{B}'$,

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix}$ equivalent to

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, and from the inverse matrix

$$\begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}^{-1} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = -\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{\mathcal{T}} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} =$$

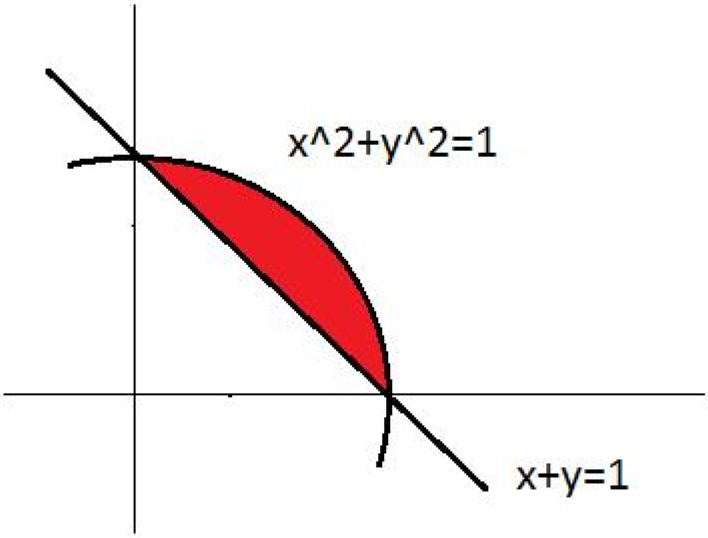
$$-\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = -\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -4 \end{pmatrix}.$$

II M 1) The equation $f(x,y) = x^4 + x^2y + e^y = 2$ defines at point $P(1,0)$ an implicity function $y = y(x)$. Calculate its first derivative $y'(1)$ and second derivative $y''(1)$.

$f(P) = 1 + 0 + e^0 = 2$, $f'_x = 4x^3 + 2xy$ and $f'_y = x^2 + e^y$, with $f'_x(P) = 4$ and $f'_y(P) = 2$. Since $f'_y(P) \neq 0$, the equation $f(x,y) = x^4 + x^2y + e^y = 2$ defines a function $y = y(x)$ with $y'(1) = -\dfrac{f'_x(P)}{f'_y(P)} = -2$. For the second order derivative we have $y''(1) = -\dfrac{f''_{x,x}(P) + 2 \cdot f''_{x,y}(P) \cdot y'(1) + f''_{y,y}(P) \cdot (y'(1))^2}{f'_y(P)} = -\dfrac{f''_{x,x}(P) - 4 \cdot f''_{x,y}(P) + 4 \cdot f''_{y,y}(P)}{2}$ because $y'(1) = -2$ and $f'_y(P) = 2$. The hessian matrix of function f is $\mathcal{H}f = \begin{bmatrix} 12x^2 + 2y & 2x \\ 2x & e^y \end{bmatrix}$ with $\mathcal{H}f(P) = \begin{bmatrix} 12 & 2 \\ 2 & 1 \end{bmatrix}$ and the second order derivative is $y''(1) = -\dfrac{12 - 4 \cdot 2 + 4 \cdot 1}{2} = -4$.

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x - y \\ \text{u.c.: } \begin{cases} x^2 + y^2 \leq 1 \\ 1 \leq x + y \end{cases} \end{cases}$.

The function f is a polynomial, continuos function, the admissible region, in red in the figure below, is a bounded and closed set, therefore f presents absolute maximum and minimum in the admissible region, constraints are qualified on any point in the border of the admissible region, the two constraints can be written as $x^2+y^2-1\leq 0$ and $1-x-y\leq 0$. The Lagrangian function is
$\mathcal{L}(x,y,\lambda)=x-y-\lambda(x^2+y^2-1)-\mu(1-x-y)$ with
$\nabla\mathcal{L}=(1-2\lambda x+\mu,\ -1-2\lambda y+\mu,\ -(x^2+y^2-4),\ -(1-x-y))$.



$I°\ CASE\ (free\ optimization)$:

$$\begin{cases}\lambda=\mu=0\\ 1=0\\ -1=0\\ x^2+y^2\leq 1\\ 1\leq x+y\end{cases}$$; system impossible.

$II°\ CASE\ (constrained\ optimization$ - $first\ constraint\ active)$:

$$\begin{cases}\lambda\neq 0,\mu=0\\ 1-2\lambda x=0\\ -1-2\lambda y=0\\ x^2+y^2=1\\ 1\leq x+y\end{cases}\Rightarrow\begin{cases}\lambda\neq 0,\mu=0\\ x=\frac{1}{2\lambda}\\ y=-\frac{1}{2\lambda}\\ x^2+y^2=1\\ 1\not\leq 0\end{cases}$$; system impossible.

$III°\ CASE\ (constrained\ optimization$ - $second\ constraint\ active)$:

$$\begin{cases}\lambda=0,\mu\neq 0\\ 1+\mu=0\\ -1+\mu=0\\ x^2+y^2\leq 1\\ 1=x+y\end{cases}\Rightarrow\begin{cases}\lambda=0,\mu\neq 0\\ \mu=-1\\ \mu=1\\ x^2+y^2\leq 1\\ 1=x+y\end{cases}$$; system impossible.

$IIII°\ CASE\ (constrained\ optimization$ - $both\ constraints\ active)$:

$$\begin{cases}\lambda\neq 0,\mu\neq 0\\ 1-2\lambda x+\mu=0\\ -1-2\lambda y+\mu=0\\ x^2+y^2=1\\ 1=x+y\end{cases}\Rightarrow\begin{cases}\lambda\neq 0,\mu\neq 0\\ 1-2\lambda x+\mu=0\\ -1-2\lambda y+\mu=0\\ x^2+(1-x)^2=1\\ y=1-x\end{cases}\Rightarrow\begin{cases}\lambda\neq 0,\mu\neq 0\\ 1-2\lambda x+\mu=0\\ -1-2\lambda y+\mu=0\\ x^2-x=0\\ y=1-x\end{cases}\Rightarrow$$

$$\begin{cases} \lambda \neq 0, \mu \neq 0 \\ 1 - 2\lambda x + \mu = 0 \\ -1 - 2\lambda y + \mu = 0 \\ x(x-1) = 0 \\ y = 1 - x \end{cases}$$
; if $x = 0$, $y = 1$ and $\lambda = \mu = -1$, otherwise if $x = 1$, $y = 0$ and $\lambda = \mu = 1$. Point $P_1(0,1)$ is point of minimum for f (λ and μ both negative), with absolute minimum $f(P_1) = -1$; point $P_2(1,0)$ is point of maximum for f (λ and μ both positive), with absolute maximum $f(P_2) = 1$.

II M 3) Given the function $f(x,y) = |x|y - x|y|$. Study if the function f is differentiable at point $O(0,0)$.

Function f is differentiable at point $O(0,0)$ if exist real numbers a and b such that $\lim_{(x,y)\to(0,0)} \dfrac{f(x,y) - f(0,0) - (ax+by)}{\sqrt{x^2+y^2}} = 0$. Using polar coordinates we have

$$\lim_{\rho\to 0} \frac{|\rho\, cos\,\theta|\,\rho\, sin\,\theta - \rho\, cos\,\theta\,|\rho\, sin\,\theta| - (a\rho\, cos\,\theta + b\rho\, sin\,\theta)}{\sqrt{(\rho\, cos\,\theta)^2 + (\rho\, sin\,\theta)^2}} =$$

$$\lim_{\rho\to 0} \frac{\rho^2\,(|cos\,\theta|\, sin\,\theta - cos\,\theta\,|sin\,\theta|) - \rho(a\, cos\,\theta + b\, sin\,\theta)}{\rho} =$$

$\lim_{\rho\to 0} \rho(|cos\,\theta|\, sin\,\theta - cos\,\theta\,|sin\,\theta|) - (a\, cos\,\theta + b\, sin\,\theta)$. From the last limit we can observe that a necessary condition such that the limit is zero is $a = b = 0$ and so our limit can be written as: $\lim_{\rho\to 0} \rho(|cos\,\theta|\, sin\,\theta - cos\,\theta\,|sin\,\theta|) = 0$. To conclude the exercise we can prove that the convergence is uniformly respect θ; for this goal note that $\left|\rho(|cos\,\theta|\, sin\,\theta - cos\,\theta\,|sin\,\theta|)\right| = \rho\left||cos\,\theta|\, sin\,\theta - cos\,\theta\,|sin\,\theta|\right| \leq \rho$, convergence is uniformly.

II M 4) Given the function $f(x,y) = xe^y + ye^x$ and the unit vector $v = (cos\,\alpha, sin\,\alpha)$; knowing that at point $(0,0)$ the directional derivative $\mathcal{D}_v f(0,0) = 0$, find the two feasible values of α and compute for both these values the second order directional derivative $\mathcal{D}^{(2)}_{v,v} f(0,0)$.

$\nabla f(x,y) = (e^y + ye^x, xe^y + e^x)$, $\nabla f(0,0) = (1,1)$;
$\mathcal{D}_v f(0,0) = \nabla f(0,0) \cdot v = (1,1) \cdot (cos\,\alpha, sin\,\alpha) = cos\,\alpha + sin\,\alpha$. Putting $cos\,\alpha + sin\,\alpha = 0$ we get $sin\,\alpha = -cos\,\alpha$ and the two feasible values of α are $\dfrac{3\pi}{4}$ and $\dfrac{7\pi}{4}$. $\mathcal{D}^{(2)}_{v,v} f(0,0) = v^T \cdot \mathcal{H}f(0,0) \cdot v$; $\mathcal{H}f = \begin{bmatrix} ye^x & e^y + e^x \\ e^y + e^x & xe^y \end{bmatrix}$,
$\mathcal{H}f(0,0) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ and $\mathcal{D}^{(2)}_{v,v} f(0,0) = (cos\,\alpha, sin\,\alpha) \cdot \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \cdot \begin{pmatrix} cos\,\alpha \\ sin\,\alpha \end{pmatrix} =$
$4\, sin\,\alpha\, cos\,\alpha = 2\, sin\, 2\alpha$; for both values of α $\mathcal{D}^{(2)}_{v,v} f(0,0) = -2$.