UNIVERSITÁ DEGLI STUDI DI SIENA
Scuola di Economia e Management
A.A. 2023/24
Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 20/3/2025

I M 1) Find all the complex numbers such that $z^3 - 4iz = 0$.
$z^3 - 4iz = z(z^2 - 4i)$, put $z(z^2 - 4i) = 0$ we find the first solution $z_1 = 0$, for the second and the third solution put $z^2 - 4i = 0$ we have $z^2 = 4i$ and $z_{2,3} = \pm\sqrt{4i} = \pm 2\sqrt{i} = \pm 2\sqrt{cos\left(\frac{\pi}{2}\right) + i\,sin\left(\frac{\pi}{2}\right)} = \pm 2\left(cos\left(\frac{\pi}{4}\right) + i\,sin\left(\frac{\pi}{4}\right)\right) = \pm 2\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) = \pm\sqrt{2}(1+i)$.

I M 2) Given the linear system $\mathbb{A}\cdot\mathbb{X} = \mathbb{Y}$, with $\mathbb{A} = \begin{bmatrix} 2 & 2 & -1 & -1 \\ 0 & 2 & 0 & -1 \\ 2 & 4 & -1 & -2 \end{bmatrix}$, find all the vectors $\mathbb{Y}$ for which the system has solutions.

The complete matrix of the system is $[\mathbb{A}|\mathbb{Y}] = \begin{bmatrix} 2 & 2 & -1 & -1 & | & y_1 \\ 0 & 2 & 0 & -1 & | & y_2 \\ 2 & 4 & -1 & -2 & | & y_3 \end{bmatrix}$; we reduce the matrix by elementary operations on the raws:

$$\begin{bmatrix} 2 & 2 & -1 & -1 & | & y_1 \\ 0 & 2 & 0 & -1 & | & y_2 \\ 2 & 4 & -1 & -2 & | & y_3 \end{bmatrix} \underset{R_3 \mapsto R_3 - R_1}{} \begin{bmatrix} 2 & 2 & -1 & -1 & | & y_1 \\ 0 & 2 & 0 & -1 & | & y_2 \\ 0 & 2 & 0 & -1 & | & y_3 - y_1 \end{bmatrix}$$

$$\underset{R_3 \mapsto R_3 - R_2}{} \begin{bmatrix} 2 & 2 & -1 & -1 & | & y_1 \\ 0 & 2 & 0 & -1 & | & y_2 \\ 0 & 0 & 0 & 0 & | & y_3 - y_1 - y_2 \end{bmatrix}$$

. By the last matrix easily we know that the incomplete matrix of the system has rank equal two and by the Rochè-Capelli Theorem the system has solution if and only if the complete matrix of the system has rank equal two; matrix $[\mathbb{A}|\mathbb{Y}]$ presents rank equal two if and only if $y_3 - y_1 - y_2 = 0$.
The vectors $\mathbb{Y}$ for which the system has solutions are all the vectors of the form $\mathbb{Y} = (y_1, y_2, y_1 + y_2)$.

I M 3) Vector $V \in \mathbb{R}^2$ has coordinates $(1, -2)$ respect the basis $\mathcal{B} = \{(1,1),(0,2)\}$. Find the coordinates of vector V respect the new basis $\mathcal{B}' = \{(1,2),(0,1)\}$.
If vector V has coordinates $(1, -2)$ respect the basis $\mathcal{B}$,
$V = 1(1,1) - 2(0,2) = (1, -3)$; if (α, β) are the coordinates of V respect the basis $\mathcal{B}'$, $V = \alpha(1,2) + \beta(0,1) = (\alpha, 2\alpha + \beta)$. Put $(\alpha, 2\alpha + \beta) = (1, -3)$ we get $\alpha = 1$ and $\beta = -5$.
Alternative solution: if $(1, -2)$ are the coordinates of vector V respect the basis $\mathcal{B}$ and (α, β) are those respect the basis $\mathcal{B}'$, $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ equivalent

to $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, and from the inverse matrix

$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{\mathcal{T}} \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$.

I M 4) Given the matrix $\mathbb{A} = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}$. Calculate the eigenvalues of the matrix $\mathbb{A}^2$. Is $\mathbb{A}^2$ a diagonalizable matrix?

$\mathbb{A}^2 = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 5 \end{bmatrix}$. The characteristic polynomial of the matrix $\mathbb{A}^2$ is $P_{\mathbb{A}^2}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}^2| = \begin{vmatrix} \lambda & -2 \\ 2 & \lambda - 5 \end{vmatrix} = \lambda(\lambda-5)+4 = \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4)$. Put $P_{\mathbb{A}^2}(\lambda) = 0$ we find the two eigenvalues of matrix $\mathbb{A}^2$: $\lambda_1 = 1$, $\lambda_2 = 4$, the two eigenvalues are different, so the matrix $\mathbb{A}^2$ is diagonalizable.

Alternative solution: the characteristic polynomial of the matrix $\mathbb{A}$ is $P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda+2 & -2 \\ 2 & \lambda-3 \end{vmatrix} = (\lambda+2)(\lambda-3)+4 = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$. Put $P_{\mathbb{A}}(\lambda) = 0$ we find the two eigenvalues of matrix $\mathbb{A}$: $\lambda_1 = -1$, $\lambda_2 = 2$ and remembering that if λ is an eigenvalue of matrix $\mathbb{A}$, then λ^2 is an eigenvalue of matrix $\mathbb{A}^2$, we conclude that 1 and 4 are the two eigenvalues of matrix $\mathbb{A}^2$ and the two eigenvalues are different, so the matrix $\mathbb{A}^2$ is diagonalizable.

II M 1) The equation $f(x,y) = e^{x-2y} + cos(x+y) = 0$ defines at point $P(2\pi, \pi)$ an implicity function $y = y(x)$. Calculate the Taylor's polynomial of second order for function $y(x)$ centered on point $x_0 = 2\pi$.

$f(P) = e^{2\pi - 2\pi} + cos(2\pi + \pi) = e^0 + cos(3\pi) = 1 - 1 = 0$,
$f'_x = e^{x-2y} - sen(x+y)$ and $f'_y = -2e^{x-2y} - sen(x+y)$, with $f'_x(P) = 1$ and $f'_y(P) = -2$. Since $f'_y(P) \neq 0$, the equation $f(x,y) = e^{x-2y} + cos(x+y) = 0$ defines a function $y = y(x)$ with $y'(2\pi) = -\dfrac{f'_x(P)}{f'_y(P)} = \dfrac{1}{2}$. For the second order derivative we have

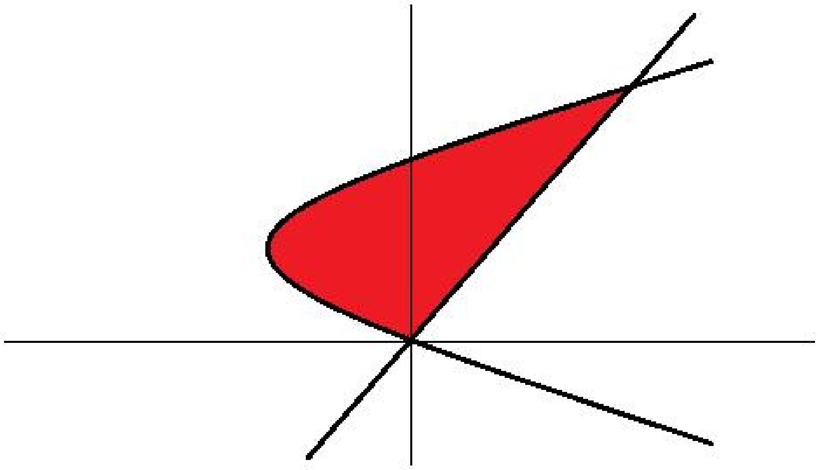
$y''(2\pi) = -\dfrac{f''_{x,x}(P) + 2 \cdot f''_{x,y}(P) \cdot y'(2\pi) + f''_{y,y}(P) \cdot (y'(2\pi))^2}{f'_y(P)} =$

$\dfrac{f''_{x,x}(P) + f''_{x,y}(P) + \frac{1}{4} \cdot f''_{y,y}(P)}{2}$ because $y'(2\pi) = \dfrac{1}{2}$ and $f'_y(P) = -2$. The second order partial derivatives are: $f''_{x,x} = e^{x-2y} - cos(x+y)$, $f''_{x,y} = -2e^{x-2y} - cos(x+y)$ and $f''_{y,y} = 4e^{x-2y} - cos(x+y)$ with $f''_{x,x}(P) = 2$, $f''_{x,y}(P) = -1$ and $f''_{y,y}(P) = 5$;

$y''(2\pi) = \dfrac{2 - 1 + \frac{1}{4} \cdot 5}{2} = \dfrac{9}{8}$. The Taylor's polynomial of second order for function $y(x)$ centered on point $x_0 = 2\pi$ is $\mathcal{P}_2(x) = \pi + \dfrac{1}{2}(x - 2\pi) + \dfrac{1}{2} \cdot \dfrac{9}{8}(x-2\pi)^2 = \dfrac{9}{16}x^2 + \dfrac{2 - 9\pi}{4}x + \dfrac{9}{4}\pi^2$.

II M 2) Solve the optimization problem $\begin{cases} \text{Max/min } f(x,y) = x - xy \\ \text{u.c.: } \begin{cases} y^2 - 2y - x \le 0 \\ x - y \le 0 \end{cases} \end{cases}$.

The function f is a polynomial, continuos function, the admissible region, in red in the figure below, is a bounded and closed set, therefore f presents absolute maximum and minimum in the admissible region, constraints are qualified on any point in the border of the admissible region. The Lagrangian function is
$\mathcal{L}(x,y,\lambda,\mu) = x - xy - \lambda(y^2 - 2y - x) - \mu(x - y)$ with
$\nabla\mathcal{L} = (1 - y + \lambda - \mu,\ -x - 2\lambda y + 2\lambda + \mu,\ -(y^2 - 2y - x),\ -(x - y))$.



$I° \; CASE$ ($free\ optimization$):

$$\begin{cases} \lambda = \mu = 0 \\ 1 - y = 0 \\ -x = 0 \\ y^2 - 2y - x \le 0 \\ x - y \le 0 \end{cases} \Rightarrow \begin{cases} \lambda = \mu = 0 \\ y = 1 \\ x = 0 \\ 1 - 2 - 0 \le 0 \\ 0 - 1 \le 0 \end{cases}$$

; point $P_1(0,1)$ is inside the admissible region, the hessian matrix of function f is $\mathcal{H}f = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\mathcal{H}_2 = -1 < 0$. P_1 is a saddle point.

$II° \; CASE$ ($constrained\ optimization$ - $first\ constraint\ active$):

$$\begin{cases} \lambda \ne 0, \mu = 0 \\ 1 - y + \lambda = 0 \\ -x - 2\lambda y + 2\lambda = 0 \\ y^2 - 2y - x = 0 \\ x - y \le 0 \end{cases} \Rightarrow \begin{cases} \lambda \ne 0, \mu = 0 \\ y = 1 + \lambda \\ x = -2\lambda^2 \\ (1+\lambda)^2 - 2(1+\lambda) + 2\lambda^2 = 0 \\ x - y \le 0 \end{cases} \Rightarrow \begin{cases} \lambda \ne 0, \mu = 0 \\ y = 1 + \lambda \\ x = -2\lambda^2 \\ 3\lambda^2 = 1 \\ x - y \le 0 \end{cases} \Rightarrow$$

$$\begin{cases} \lambda \ne 0, \mu = 0 \\ y = 1 + \lambda \\ x = -2\lambda^2 \\ \lambda = \pm\frac{1}{3}\sqrt{3} \\ x - y \le 0 \end{cases} \Rightarrow \begin{cases} \lambda \ne 0, \mu = 0 \\ y = 1 \pm \frac{1}{3}\sqrt{3} \\ x = -\frac{2}{3} \\ \lambda = \pm\frac{1}{3}\sqrt{3} \\ -\frac{2}{3} - 1 \mp \frac{1}{3}\sqrt{3} \le 0 \end{cases}$$

; both points are admissible, point $P_2\left(-\frac{2}{3}, 1 + \frac{1}{3}\sqrt{3}\right)$ has $\lambda = +\frac{1}{3}\sqrt{3} > 0$, P_2 is a candidate for maximum, while point $P_3\left(-\frac{2}{3}, 1 - \frac{1}{3}\sqrt{3}\right)$ has $\lambda = -\frac{1}{3}\sqrt{3} < 0$, P_3 is a candidate for minimum.

$III° \; CASE$ ($constrained\ optimization$ - $second\ constraint\ active$):

$$\begin{cases} \lambda = 0, \mu \neq 0 \\ 1 - y - \mu = 0 \\ -x + \mu = 0 \\ y^2 - 2y - x \leq 0 \\ x - y = 0 \end{cases} \Rightarrow \begin{cases} \lambda = 0, \mu \neq 0 \\ y = 1 - \mu \\ x = \mu \\ y^2 - 2y - x \leq 0 \\ \mu - 1 + \mu = 0 \end{cases} \Rightarrow \begin{cases} \lambda = 0, \mu \neq 0 \\ y = 1 - \mu \\ x = \mu \\ y^2 - 2y - x \leq 0 \\ 2\mu = 1 \end{cases} \Rightarrow$$

$$\begin{cases} \lambda = 0, \mu \neq 0 \\ y = \frac{1}{2} \\ x = \frac{1}{2} \\ \left(\frac{1}{2}\right)^2 - 1 - \frac{1}{2} \leq 0 \\ \mu = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} \lambda = 0, \mu \neq 0 \\ y = \frac{1}{2} \\ x = \frac{1}{2} \\ -\frac{5}{4} \leq 0 \\ \mu = \frac{1}{2} \end{cases}$$

; point $P_4\left(\frac{1}{2}, \frac{1}{2}\right)$ is admissible and has $\mu = \frac{1}{2} > 0$, P_4 is a candidate for maximum.

$IIII^\circ\ CASE$ ($constrained\ optimization$ - $both\ constraints\ active$):

$$\begin{cases} \lambda \neq 0, \mu \neq 0 \\ 1 - y + \lambda - \mu = 0 \\ -x - 2\lambda y + 2\lambda + \mu = 0 \\ y^2 - 2y - x = 0 \\ x - y = 0 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0, \mu \neq 0 \\ 1 - y + \lambda - \mu = 0 \\ -y - 2\lambda y + 2\lambda + \mu = 0 \\ y^2 - 2y - y = 0 \\ y = x \end{cases} \Rightarrow$$

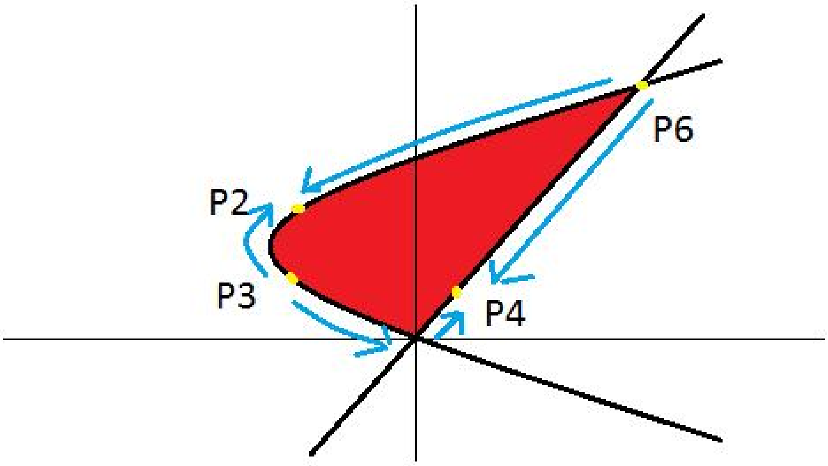
$$\begin{cases} \lambda \neq 0, \mu \neq 0 \\ 1 - y + \lambda - \mu = 0 \\ -y - 2\lambda y + 2\lambda + \mu = 0 \\ y(y-3) = 0 \\ y = x \end{cases}$$

; if $y = 0, x = 0,\ \lambda = -\frac{1}{3}$ and $\mu = \frac{2}{3}$, point $P_5(0,0)$ is admissible but isn't a maximum and isn't a minimum because λ and μ have opposite sign; if $y = 3, x = 3,\ \lambda = -\frac{5}{3}$ and $\mu = -\frac{11}{3}$, point $P_6(3,3)$ is admissible and is a candidate for minimum (λ and μ both negative).

Now we study the function f along the border:

1. if $y^2 - 2y - x = 0, f(x,y) = f(y^2 - 2y, y) = y^2 - 2y - (y^2 - 2y)y = -y^3 + 3y^2 - 2y = g(y), g'(y) = -3y^2 + 6y - 2, g'(y) \geq 0$ if and only if $3y^2 - 6y + 2 \leq 0$, that is true if $1 - \frac{1}{3}\sqrt{3} \leq y \leq 1 + \frac{1}{3}\sqrt{3}$, along the upper border function f is decresing from $0 \leq y \leq 1 - \frac{1}{3}\sqrt{3}$, increasing from $1 - \frac{1}{3}\sqrt{3} \leq y \leq 1 + \frac{1}{3}\sqrt{3}$ and again decreasing from $1 + \frac{1}{3}\sqrt{3} \leq y \leq 3$;
2. if $y - x = 0, f(x,y) = f(y,y) = y - y^2 = h(y), h'(y) = 1 - 2y, h'(y) \geq 0$ if and only if $y \leq \frac{1}{2}$, along the lower border function f is incresing from $0 \leq y \leq \frac{1}{2}$, increasing from $\frac{1}{2} \leq y \leq 3$.

The behavior of function f along the border is depicted in the graphic in the next page.



$f(P_2) = -\frac{2}{3} + \frac{2}{3}\left(1 + \frac{1}{3}\sqrt{3}\right) = \frac{2}{9}\sqrt{3}$, $f(P_4) = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$; f disclose absolute maximum in point P_2, P_4 is point of local maximum.

$f(P_3) = -\frac{2}{3} + \frac{2}{3}\left(1 - \frac{1}{3}\sqrt{3}\right) = -\frac{2}{9}\sqrt{3}$, $f(P_6) = 3 - 3 \cdot 3 = -6$; f disclose absolute minimum in point P_6, P_3 is point of local minimum.

II M 3) Given the function $f(x,y) = |xy| - xy$. Study if the function f is differentiable at point $O(0,0)$.

Function f is differentiable at point $O(0,0)$ if exist real numbers a and b such that $\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - (ax+by)}{\sqrt{x^2+y^2}} = 0$. Using polar coordinates we have

$$\lim_{\rho\to 0} \frac{|\rho\cos\theta \cdot \rho\sin\theta| - \rho\cos\theta \cdot \rho\sin\theta - (a\rho\cos\theta + b\rho\sin\theta)}{\sqrt{(\rho\cos\theta)^2 + (\rho\sin\theta)^2}} =$$

$$\lim_{\rho\to 0} \frac{\rho^2\left(|\cos\theta \cdot \sin\theta| - \cos\theta \cdot \sin\theta\right) - \rho(a\cos\theta + b\sin\theta)}{\rho} =$$

$\lim_{\rho\to 0} \rho(|\cos\theta \cdot \sin\theta| - \cos\theta \cdot \sin\theta) - (a\cos\theta + b\sin\theta)$. From the last limit we can observe that a necessary condition such that the limit is zero is $a = b = 0$ and so our limit can be written as: $\lim_{\rho\to 0} \rho(|\cos\theta \cdot \sin\theta| - \cos\theta \cdot \sin\theta) = 0$. To conclude the exercise we can prove that the convergence is uniformly respect θ; for this goal note that $\left|\rho(|\cos\theta \cdot \sin\theta| - \cos\theta \cdot \sin\theta)\right| =$
$\rho\left|\frac{1}{2}|\sin 2\theta| - \frac{1}{2}\sin 2\theta\right| \le \rho|\sin 2\theta| \le \rho$, convergence is uniformly.

II M 4) Given the two functions, $g: \mathbb{R} \to \mathbb{R}^2$, with $g(t) = (at, bt)$ and $f: \mathbb{R}^2 \to \mathbb{R}^2$, with $f(x,y) = (x+y, x-y)$, and consider the composite function $h(t) = f(g(t))$. Find the values of the parameters a and b knowing that $h'(1) = (1,1)$, and calculate the equation of tangent line at the graphic of h at point $t_0 = -1$.

$h'(1) = \mathcal{J}f(g(1)) \cdot g'(1) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a-b \end{pmatrix}$; put $\begin{pmatrix} a+b \\ a-b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we find $a = 1$ and $b = 0$. The equation of tangent line, in parametric form, at the graphic of h at point $t_0 = -1$ is $r(t) = h(-1) + h'(-1) \cdot (t+1)$;
$h(-1) = f(g(-1)) = f(-1, 0) = (-1, -1)$,

$h'(-1) = \mathcal{J}f(g(-1)) \cdot g'(-1) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and follow $r(t) = (-1, -1) + (1,1) \cdot (t+1) = (t,t)$.
Alternative solution: $h(t) = f(g(t)) = f(at, bt) = (at + bt, at - bt)$, $h'(t) = (a+b, a-b)$; put $(a+b, a-b) = (1,1)$ we get $a = 1$ and $b = 0$, the remaining follows as above.