UNIVERSITÁ DEGLI STUDI DI SIENA
Scuola di Economia e Management
A.A. 2024/25
Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 6/6/2025

I M 1) Find the complex number z such that it's satisfied the following equality:
$\dfrac{(2-i)^2-(2+i)^2}{z-2i}+2i=0$.
$\dfrac{(2-i)^2-(2+i)^2}{z-2i}+2i=\dfrac{(2-i)^2-(2+i)^2+2i(z-2i)}{z-2i}=$
$\dfrac{4-4i+i^2-(4+4i+i^2)+2iz-4i^2}{z-2i}=\dfrac{-8i+2iz+4}{z-2i}$; put $-8i+2iz+4=0$
we find $z=\dfrac{8i-4}{2i}=\dfrac{4i-2}{i}=\dfrac{4i-2}{i}\cdot\dfrac{i}{i}=\dfrac{4i^2-2i}{i^2}=\dfrac{-4-2i}{-1}=4+2i$.

I M 2) Given the matrix $\mathbb{A}=\begin{bmatrix}1&0&0\\0&-2&0\\2&0&1\end{bmatrix}$. Study if the matrix is diagonalizable and for eigenvalue $\lambda=-2$ calculate a basis for the associated eigenspace.

To study the diagonalizability of $\mathbb{A}$ we start with the calculus of the characteristic polynomial of the matrix; $P_{\mathbb{A}}(\lambda)=|\lambda\mathbb{I}-\mathbb{A}|=\begin{vmatrix}\lambda-1&0&0\\0&\lambda+2&0\\-2&0&\lambda-1\end{vmatrix}=$
$(\lambda-1)^2(\lambda+2)$. Putting $P_{\mathbb{A}}(\lambda)=0$ we find the three eigenvalues of matrix $\mathbb{A}$: $\lambda_{1,2}=1$, $\lambda_3=-2$, the eigenvalue 1 has algebraic multiplicity equal two. To verify if the matrix is diagonalizable, we must find the geometric multiplicity of eigenvalue 1, for our goal we calculate the rank of matrix $\mathbb{I}-\mathbb{A}=\begin{bmatrix}0&0&0\\0&3&0\\-2&0&0\end{bmatrix}$, it's easy note that from matrix $\mathbb{A}$ we can define a principal minor of order 2, $\begin{bmatrix}0&3\\-2&0\end{bmatrix}$ with determinant different from 0 and matrix $\mathbb{I}-\mathbb{A}$ has the first raw null, thus $Rank(\mathbb{I}-\mathbb{A})=2$ and the geometric multiplicity of eigenvalue 1 is one. The matrix isn't diagonalizable. For a basis for the eigenspace associated at the eigenvalue $\lambda=-2$ we consider the matrix
$-2\mathbb{I}-\mathbb{A}=\begin{bmatrix}-3&0&0\\0&0&0\\-2&0&-3\end{bmatrix}$; a generic vector that belongs to this eigenspace is a vector (x,y,z) such that $\begin{bmatrix}-3&0&0\\0&0&0\\-2&0&-3\end{bmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-3z\\0\\-2x-3z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow$
$x=z=0$. Thus a generic vector that belongs to the eigenspace is a vector $(0,y,0)=y(0,1,0)$, and a basis for the eigenspace is the set $\mathcal{B}_{ES_{\lambda=-2}}=\{(0,1,0)\}$.

I M 3) Given a linear map $F\colon\mathbb{R}^3\to\mathbb{R}^4$, with
$F(x_1,x_2,x_3)=(x_1+x_2,x_2+x_3,x_1+x_2+x_3,-x_1-x_2-x_3)$. Find the matrix

associated with the linear map, calculate the dimentions of both, kernel and immage of F, and find a basis for the image.
A generic element of the image of linear map is the vector:
$\begin{pmatrix} x_1+x_2 \\ x_2+x_3 \\ x_1+x_2+x_3 \\ -x_1-x_2-x_3 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and easily we get that the matrix associated with the linear map is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$. Now we reduce the matrix by elementary operations on its lines:
$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{matrix} R_3 \mapsto R_3 - R_1 \\ R_4 \mapsto R_4 + R_3 \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$; the matrix shows full rank, thus the dimention of the image is 3 and by the Rank-Nullity Theorem $dim(Ker(F)) = dim(\mathbb{R}^3) - dim(Ima(F)) = 3 - 3 = 0$. For a basis of the image note that a generic vector that belongs in the image $(x_1+x_2, x_2+x_3, x_1+x_2+x_3, -x_1-x_2-x_3)$ has the fourth component equal the opposite of the third component and we can conclude that all the vectors in the image have form $(y_1, y_2, y_3, -y_3) = y_1(1,0,0,0) + y_2(0,1,0,0) + y_3(0,0,1,-1)$; a basis for the image is the set $\mathcal{B}_{Ima(F)} = \{(1,0,0,0), (0,1,0,0), (0,0,1,-1)\}$.

I M 4) Consider the matrix $\mathbb{U} = \begin{bmatrix} k & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & k \end{bmatrix}$. Knowing that the matrix $\mathbb{U}$ is an horthogonal matrix, calculate the value of k and find the eigenvalues of matrix $\mathbb{U}$.
Remember that a matrix $\mathbb{U}$ is a horthogonal matrix if and only if $\mathbb{U} \cdot \mathbb{U}^T = \mathbb{U}^T \cdot \mathbb{U} = \mathbb{I}$, an identity matrix. Also matrix $\mathbb{U}$ is symmetrical, so $\mathbb{U}^T = \mathbb{U}$ and $\mathbb{U} \cdot \mathbb{U}^T = \mathbb{U}^T \cdot \mathbb{U} = \mathbb{U} \cdot \mathbb{U}$.
$\mathbb{U} \cdot \mathbb{U} = \begin{bmatrix} k & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & k \end{bmatrix} \cdot \begin{bmatrix} k & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & k \end{bmatrix} = \begin{bmatrix} 1+k^2 & 0 & 2k \\ 0 & 1 & 0 \\ 2k & 0 & 1+k^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ if and only if $k = 0$; $\mathbb{U} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. To find the eigenvalues of matrix $\mathbb{U}$ we calculate the characteristic polynomial of the matrix: $P_{\mathbb{U}}(\lambda) = |\lambda\mathbb{I} - \mathbb{U}| = \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda \end{vmatrix} =$
$(\lambda-1)\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = (\lambda-1)(\lambda^2-1) = (\lambda-1)^2(\lambda+1)$. Putting $P_{\mathbb{U}}(\lambda) = 0$ we find the three eigenvalues of matrix $\mathbb{U}$: $\lambda_{1,2} = 1$, $\lambda_3 = -1$.

II M 1) With the equation $f(x,y,z) = xyz + 2xy^2 + 2xz^2 + 3yz = 0$ we can defined in a neighbourhood of point $P(1,-1,1)$ a function in implicit form. Which type of implicit function can we define? Calculate its first order derivatives.
$f(P) = -1 + 2 + 2 - 3 = 0$, condition is satisfied in point P.
$\nabla f = \left(yz + 2y^2 + 2z^2, xz + 4xy + 3z, xy + 4xz + 3y\right)$, $\nabla f(P) = (3,0,0)$.

In point P only $f'_x(P) \neq 0$, thus the proposed condition defines a implicit function $(y,z) \mapsto x(y,z)$ with $x'_y(-1,1) = -\dfrac{f'_y(P)}{f'_x(P)} = 0$ and $x'_z(-1,1) = -\dfrac{f'_z(P)}{f'_x(P)} = 0$. $(-1,1)$ is a critical point for function $x(y,z)$.

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = 3y - x \\ \text{u.c.: } x^2 + y^2 \leq 4 \end{cases}$.

The function f is a polynomial, continuos function, the admissible region is a disk with center $(0,0)$ and radius 2, a bounded and closed set, therefore f presents absolute maximum and minimum in the admissible region, constraint is qualified on any point in the circumference $x^2 + y^2 = 4$. The Lagrangian function is
$\mathcal{L}(x,y,\lambda) = 3y - x - \lambda(x^2 + y^2 - 4)$ with
$\nabla\mathcal{L} = (-1 - 2\lambda x, 3 - 2\lambda y, -(x^2 + y^2 - 4))$.

$I° \; CASE$ ($free\ optimization$):

$\begin{cases} \lambda = 0 \\ -1 = 0 \\ 3 = 0 \\ x^2 + y^2 \leq 4 \end{cases}$; System impossible.

$II° \; CASE$ ($constrained\ optimization$):

$$\begin{cases} \lambda \neq 0 \\ -1 - 2\lambda x = 0 \\ 3 - 2\lambda y = 0 \\ x^2 + y^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x = -\frac{1}{2\lambda} \\ y = \frac{3}{2\lambda} \\ \left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x = -\frac{1}{2\lambda} \\ y = \frac{3}{2\lambda} \\ \frac{10}{4\lambda^2} = 4 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x = -\frac{1}{2\lambda} \\ y = \frac{3}{2\lambda} \\ \lambda^2 = \frac{10}{16} \end{cases} \Rightarrow$$

$\begin{cases} \lambda \neq 0 \\ x = \mp \frac{1}{5}\sqrt{10} \\ y = \pm \frac{3}{5}\sqrt{10} \\ \lambda = \pm \frac{1}{4}\sqrt{10} \end{cases}$. Two critical points, $P_1 = \left(-\frac{1}{5}\sqrt{10}, +\frac{3}{5}\sqrt{10}\right)$ the unique candidate for maximum ($\lambda > 0$), and $P_2 = \left(+\frac{1}{5}\sqrt{10}, -\frac{3}{5}\sqrt{10}\right)$, the unique candidate for minimum ($\lambda < 0$). $Max f = f(P_1) = 3\left(\frac{3}{5}\sqrt{10}\right) - \left(-\frac{1}{5}\sqrt{10}\right) = 2\sqrt{10}$, $Min f = f(P_2) = 3\left(-\frac{3}{5}\sqrt{10}\right) - \left(\frac{1}{5}\sqrt{10}\right) = -2\sqrt{10} = -Max f$.

II M 3) Given the function $f(x,y) = |2xy|$. Study if the function f is differentiable at point $O(0,0)$.

Function f is differentiable at point $O(0,0)$ if exist real numbers a and b such that $\lim\limits_{(x,y) \to (0,0)} \dfrac{f(x,y) - f(0,0) - (ax + by)}{\sqrt{x^2 + y^2}} = 0$. Using polar coordinates we have

$$\lim_{\rho \to 0} \frac{|2\,\rho\cos\theta \cdot \rho\sin\theta| - (a\rho\cos\theta + b\rho\sin\theta)}{\sqrt{(\rho\cos\theta)^2 + (\rho\sin\theta)^2}} =$$

$$\lim_{\rho \to 0} \frac{2\,\rho^2\,|\cos\theta \cdot \sin\theta| - \rho(a\cos\theta + b\sin\theta)}{\rho} =$$

$\lim\limits_{\rho \to 0} 2\rho\,|\cos\theta \cdot \sin\theta| - (a\cos\theta + b\sin\theta)$. From the last limit we can observe that a necessary condition such that the limit is zero is $a = b = 0$ and so our limit can be written as: $\lim\limits_{\rho \to 0} 2\rho\,|\cos\theta \cdot \sin\theta| = 0$. To conclude the exercise we can prove that the convergence is uniformly respect θ; for this goal note that $\big|2\rho\,|\cos\theta \cdot \sin\theta|\big| = \rho\,|2 \cdot \sin\,\theta \cdot \cos\theta| = \rho\,|\sin 2\theta| \leq \rho$, convergence is uniformly.

II M 4) Given the function $f(x,y,z) = x^3 - y^3 - z^3 - 3x + 12yz$, find its critical points and study their nature.

$\nabla f = \left(3x^2 - 3,\ -3y^2 + 12z,\ -3z^2 + 12y\right)$.

FOC:

$$\begin{cases} 3x^2 - 3 = 0 \\ -3y^2 + 12z = 0 \\ -3z^2 + 12y = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 = 3 \\ 12z = 3y^2 \\ -3z^2 + 12y = 0 \end{cases} \Rightarrow \begin{cases} x^2 = 1 \\ z = \frac{1}{4}y^2 \\ -3\left(\frac{1}{4}y^2\right)^2 + 12y = 0 \end{cases} \Rightarrow$$

$$\begin{cases} x = \pm 1 \\ z = \frac{1}{4}y^2 \\ -\frac{3}{16}y^4 + 12y = 0 \end{cases} \Rightarrow \begin{cases} x = \pm 1 \\ z = \frac{1}{4}y^2 \\ \frac{192y - 3y^4}{16} = 0 \end{cases} \Rightarrow \begin{cases} x = \pm 1 \\ z = \frac{1}{4}y^2 \\ 3y(64 - y^3) = 0 \end{cases}$$; if $y = 0$ then $z = 0$,

otherwise if $64 - y^3 = 0$ we get $y = 4$ and $z = 4$. Four critical points:
$P_{1,2} = (\pm 1, 0, 0)$ and $P_{3,4} = (\pm 1, 4, 4)$.

SOC:

$$\mathcal{H}_f = \begin{bmatrix} 6x & 0 & 0 \\ 0 & -6y & 12 \\ 0 & 12 & -6z \end{bmatrix}$$, with $\mathcal{H}_f^1 = 6x$, $\mathcal{H}_f^2 = \begin{vmatrix} 6x & 0 \\ 0 & -6y \end{vmatrix} = -36xy$ and

$$\mathcal{H}_f^3 = \begin{vmatrix} 6x & 0 & 0 \\ 0 & -6y & 12 \\ 0 & 12 & -6z \end{vmatrix} = 6x \begin{vmatrix} -6y & 12 \\ 12 & -6z \end{vmatrix} = 6x(36yz - 144) = 216x(yz - 4).$$

For the sign sequence in the four points we get:
$\mathcal{H}_f^1(P_1) = 6 > 0$, $\mathcal{H}_f^2(P_1) = 0$, $\mathcal{H}_f^3(P_1) = -864 < 0$; P_1 saddle point, because $\mathcal{H}_f^1(P_1)$ and $\mathcal{H}_f^3(P_1)$ have opposite sign;
$\mathcal{H}_f^1(P_2) = -6 < 0$, $\mathcal{H}_f^2(P_2) = 0$, $\mathcal{H}_f^3(P_2) = 864 > 0$; P_2 saddle point;
$\mathcal{H}_f^1(P_3) = 6 > 0$, $\mathcal{H}_f^2(P_3) = -144 < 0$, $\mathcal{H}_f^3(P_3) = 2592 > 0$; P_3 saddle point, because $\mathcal{H}_f^2(P_3)$ is negative;
$\mathcal{H}_f^1(P_4) = -6 < 0$, $\mathcal{H}_f^2(P_4) = 144 > 0$, $\mathcal{H}_f^3(P_4) = -2592 < 0$; P_4 point of maximum, because the odd principal minors are negative and the unique even principal minor is positive.