UNIVERSITÁ DEGLI STUDI DI SIENA
Scuola di Economia e Management
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Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 4/7/2025

I M 1) Given the complex number $z = 1 + i$, calculate the square roots of complex number z^3.
From $z = 1 + i$ we get $z^3 = (1 + i)^3 = 1 + 3i + 3i^2 + i^3 = 1 + 3i - 3 - i = -2 + 2i$, the module of z^3 is $\rho = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ while its argument is $\theta = \pi + arctan\left(\frac{2}{-2}\right) = \pi + arctan(-1) = \pi - arctan\,1 = \pi - \frac{\pi}{4} = \frac{3}{4}\pi$; thus $z^3 = 2\sqrt{2}\left(cos\frac{3}{4}\pi + i\,sin\frac{3}{4}\pi\right)$. For the square roots of z^3 we use the classic formula:

$$\sqrt{z^3} = \sqrt{2\sqrt{2}\left(cos\frac{3}{4}\pi + i\,sin\frac{3}{4}\pi\right)} =$$

$$\sqrt[4]{8}\left(cos\left(\frac{3\pi/4 + 2k\pi}{2}\right) + i\,sin\left(\frac{3\pi/4 + 2k\pi}{2}\right)\right) =$$

$\sqrt[4]{8}\left(cos\left(\frac{3}{8}\pi + k\pi\right) + i\,sin\left(\frac{3}{8}\pi + k\pi\right)\right)$ with $k = 0, 1$. The two root are:

$k = 0 \to z_0 = \sqrt[4]{8}\left(cos\frac{3}{8}\pi + i\,sin\frac{3}{8}\pi\right) = \sqrt[4]{3 - 2\sqrt{2}} + i\sqrt[4]{3 + 2\sqrt{2}}$;

$k = 1 \to z_1 = \sqrt[4]{8}\left(cos\frac{11}{8}\pi + i\,sin\frac{11}{8}\pi\right) = -\sqrt[4]{3 - 2\sqrt{2}} - i\sqrt[4]{3 + 2\sqrt{2}} = -z_0$.

I M 2) Given the matrix $\mathbb{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & k & 2 \\ 0 & 2 & k \end{bmatrix}$ with k a real parameter. Study, varing the parameter k, if the matrix is diagonalizable.
To study the diagonalizability of $\mathbb{A}$ we start with the calculus of the characteristic polynomial of the matrix; $P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ 0 & \lambda - k & -2 \\ 0 & -2 & \lambda - k \end{vmatrix} =$
$(\lambda - 1)\begin{vmatrix} \lambda - k & -2 \\ -2 & \lambda - k \end{vmatrix} = (\lambda - 1)\left((\lambda - k)^2 - 4\right)$. Putting $P_{\mathbb{A}}(\lambda) = 0$ we find the three eigenvalues of matrix $\mathbb{A}$: for $\lambda - 1 = 0$ we have $\lambda_1 = 1$ and for $(\lambda - k)^2 - 4 = 0$ we get $(\lambda - k)^2 = 4$ and from it $\lambda_{2,3} = k \pm \sqrt{4} = k \pm 2$; matrix $\mathbb{A}$ presents multiple eigenvalue if $k + 2 = 1$ ($k = -1$) or $k - 2 = 1$ ($k = 3$). Matrix $\mathbb{A}$ is diagonalizable if $k \neq -1$ and $k \neq 3$, the two cases $k = -1$ or $k = 3$ must be studied separately; if $k = -1$ and $\lambda = 1$, matrix $\mathbb{I} - \mathbb{A} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix}$ has rank equal 2 and the geometric multiplicity of eigenvalue $\lambda = 1$ is 1, matrix isn't diagonalizable; if $k = 3$ and

$\lambda = 1$, matrix $\mathbb{I} - \mathbb{A} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$ has rank equal 1 and the geometric multiplicity of eigenvalue $\lambda = 1$ is 2, matrix is diagonalizable. In conclusion the proposed matrix is diagonalizable if and only if $k \neq -1$.

I M 3) Given a linear map $F: \mathbb{R}^3 \to \mathbb{R}^3$, with $F(x_1, x_2, x_3) = (x_1 + x_2 + x_3, kx_1 + kx_2, x_2 + kx_3)$, where k is a real parameter; we know that the image of vector $(1,1,1)$ is the vector $(3,4,3)$. Find the value of the parameter k and calculate a basis for the image of such linear map.

$F(1,1,1) = (3, 2k, 1+k)$, put $(3, 2k, 1+k) = (3,4,3)$ easily we find $k = 2$. To find a basis for the image we can note that any element of the image $\mathbb{Y} = (y_1, y_2, y_3)$ is a linear combination of columns of matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$,

$(y_1, y_2, y_3) = x_1(1,2,0) + x_2(1,2,1) + x_3(1,0,2)$, and the determinant of the matrix is $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} = 4 - 4 + 2 = 2 \neq 0$; thus the columns of the matrix are linear indipendent vectors and a basis for the image is the set of the three vectors: $\mathcal{B}_{Ima(F)} = \{(1,2,0), (1,2,1), (1,0,2)\}$.

Alternative solution to find a basis for image (by the matrix associated to the linear application): from $F(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + 2x_2, x_2 + 2x_3)$ easily we find the matrix associated to the linear application $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$; now we reduce the matrix by elementary operations on its lines: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} R_2 \mapsto R_2 - 2 \cdot R_1$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix} R_2 \circlearrowleft R_3 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$. The matrix associated has full rank thus the image of F is its codomain, $\mathbb{R}^3$; and to find a basis we can take the set of the canonical vectors: $\mathcal{B}_{Ima(F)} = \{(1,0,0), (0,1,0), (0,0,1)\}$.

I M 4) Study, varing the real parameters k and m, the number of solutions of the linear system: $\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + 3x_2 + 2x_3 = 2 \\ kx_1 + x_2 + x_3 = m \end{cases}$.

To solve the exercise we use the Rouché-Capelli Theorem in matrix form; we start writing the system in matrix form: $\begin{bmatrix} x_1 & 2x_2 & x_3 & | & 1 \\ 2x_1 & 3x_2 & 2x_3 & | & 2 \\ kx_1 & x_2 & x_3 & | & m \end{bmatrix}$, now we reduce the system by elementary operations on its rows:

$\begin{bmatrix} x_1 & 2x_2 & x_3 & | & 1 \\ 2x_1 & 3x_2 & 2x_3 & | & 2 \\ kx_1 & x_2 & x_3 & | & m \end{bmatrix} \begin{matrix} R_2 \mapsto R_2 - 2 \cdot R_1 \\ R_3 \mapsto R_3 - k \cdot R_1 \end{matrix}$

$\begin{bmatrix} x_1 & 2x_2 & x_3 & | & 1 \\ 0 & -x_2 & 0 & | & 0 \\ 0 & (1-2k)x_2 & (1-k)x_3 & | & m-k \end{bmatrix} R_3 \mapsto R_3 + (1-2k) \cdot R_2$

$\begin{bmatrix} x_1 & 2x_2 & x_3 & | & 1 \\ 0 & -x_2 & 0 & | & 0 \\ 0 & 0 & (1-k)x_3 & | & m-k \end{bmatrix}$. From the last matrix we observe that if $k \neq 1$ the complete and the incomplete matrices have the same rank 3, in this case system has only one solution indipendently from m; if $k = 1$ we rewrite the matrix and we substitute the parameter: $\begin{bmatrix} x_1 & 2x_2 & x_3 & | & 1 \\ 0 & -x_2 & 0 & | & 0 \\ 0 & 0 & 0 & | & m-1 \end{bmatrix}$; in this case the incomplete matrix has rank 2 while the complete matrix has rank 2 if and only if $m = 1$, with $k = m = 1$ the system has ∞^1 solutions (one degree of freedom in the choice of the unknows). At the end we can summarize that the number of solution of the system is: 1 if $k \neq 1$; ∞^1 if $k = m = 1$ and 0 otherwise.

II M 1) With the equation $f(x, y, z) = x^3y^2z - xy^2z^3 = 0$ we can define in a neighbourhood of point $P(1, 1, 1)$ in implicit form a function $(x, y) \mapsto z(x, y)$. Calculate its first order derivatives and write the equation of tangent plane at the graphic of function $z(x, y)$.

$f(P) = 1 - 1 = 0$, condition is satisfied in point P.

$\nabla f = \left(3x^2y^2z - y^2z^3, 2x^3yz - 2xyz^3, x^3y^2 - 3xy^2z^2\right)$, $\nabla f(P) = (2, 0, -2)$.

In point P $f'_z(P) \neq 0$, thus the proposed condition defines a implicit function $(x, y) \mapsto z(x, y)$ with $z'_x(1, 1) = -\dfrac{f'_x(P)}{f'_z(P)} = 1$ and $z'_y(1, 1) = -\dfrac{f'_y(P)}{f'z(P)} = 0$. The equation of tangent plane at the graphic of function $z(x, y)$ in point P is

$z - z(1, 1) = z'_x(1, 1) \cdot (x - 1) + z'_y(1, 1) \cdot (y - 1) \Rightarrow$

$z - 1 = 1 \cdot (x - 1) + 0 \cdot (y - 1) \Rightarrow z - 1 = x - 1$ or $x - z = 0$.

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x, y) = x^2 - y^2 \\ \text{u.c.: } x^2 + y^2 \leq 1 \end{cases}$.

The function f is a polynomial, continuos function, the admissible region is a disk with center $(0, 0)$ and radius 1, a bounded and closed set, therefore f presents absolute maximum and minimum in the admissible region, constraint is qualified in any point in the circumference $x^2 + y^2 = 1$. The Lagrangian function is

$\mathcal{L}(x, y, \lambda) = x^2 - y^2 - \lambda(x^2 + y^2 - 1)$ with

$\nabla\mathcal{L} = (2x - 2\lambda x, -2y - 2\lambda y, -(x^2 + y^2 - 1))$.

$I^\circ\ CASE\ (free\ optimization)$:

$\begin{cases} \lambda = 0 \\ 2x = 0 \\ -2y = 0 \\ x^2 + y^2 \leq 1 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ x = 0 \\ y = 0 \\ 0 \leq 1 \end{cases}$; $\mathcal{H}f = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$, $|\mathcal{H}f| = -4 < 0$; Point $O(0, 0)$ is a saddle point.

$II^\circ\ CASE\ (constrained\ optimization)$:

$\begin{cases} \lambda \neq 0 \\ 2x - 2\lambda x = 0 \\ -2y - 2\lambda y = 0 \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ 2x(1 - \lambda) = 0 \\ -2y(1 + \lambda) = 0 \\ x^2 + y^2 = 1 \end{cases}$.

If $x = 0$, $y = \pm 1$ and $\lambda = -1$, two critical points, $P_{1,2} = (0, \pm 1)$ candidates for minimum ($\lambda < 0$).

If $y = 0$, $x = \pm 1$ and $\lambda = 1$, two critical points, $P_{3,4} = (\pm 1, 0)$ candidates for maximum ($\lambda > 0$).

$f(P_{1,2}) = -(\pm 1)^2 = -1 = Minf$.

$f(P_{3,4}) = (\pm 1)^2 = 1 = Maxf.$

II M 3) Given the function $f(x,y) = x^3 - y^3 + 2xy$, find its critical points and study their nature.

$\nabla f = \left(3x^2 + 2y,\ -3y^2 + 2x\right).$

FOC:

$\begin{cases} 3x^2 + 2y = 0 \\ -3y^2 + 2x = 0 \end{cases} \Rightarrow \begin{cases} y = -\frac{3}{2}x^2 \\ -3\left(-\frac{3}{2}x^2\right)^2 + 2x = 0 \end{cases} \Rightarrow \begin{cases} y = -\frac{3}{2}x^2 \\ -\frac{27}{4}x^4 + 2x = 0 \end{cases} \Rightarrow$

$\begin{cases} y = -\frac{3}{2}x^2 \\ x\left(-\frac{27}{4}x^3 + 2\right) = 0 \end{cases}$; if $x = 0$ then $y = 0$, otherwise if $-\dfrac{27}{4}x^3 + 2 = 0$ then

$x = \sqrt[3]{\dfrac{8}{27}} = \dfrac{2}{3}$ and $y = -\dfrac{2}{3}$. Two critical points $O = (0,0)$ and $P = \left(\dfrac{2}{3},\ -\dfrac{2}{3}\right)$.

SOC:

$\mathcal{H}_f = \begin{bmatrix} 6x & 2 \\ 2 & -6y \end{bmatrix}$, with $|\mathcal{H}_f| = \begin{vmatrix} 6x & 2 \\ 2 & -6y \end{vmatrix} = -36xy - 4;\ |\mathcal{H}_f(O)| = -4 < 0,$

O is a saddle point for function $f(x,\ y)$; $|\mathcal{H}_f(P)| = 12 > 0$ and $f''_{xx}(P) = 4 > 0$, P is a point of minimum for function $f(x,\ y)$.

II M 4) Given the function $f(x,y) = x^2 - 3xy + 2y^2$ and the two unit vectors $v = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ and $w = \left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$; knowing that at point P the directional derivatives are $\mathcal{D}_v f(P) = \sqrt{2}$ and $\mathcal{D}_w f(P) = 0$, find the point P.

Function f is a polynomial, a differentiable function at any point $(x,\ y)$,

$\nabla f = (2x - 3y,\ -3x + 4y)$ and the two direction derivatives at point P are :

$\mathcal{D}_v f(P) = \nabla f \cdot v = (2x - 3y,\ -3x + 4y) \cdot \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) =$

$\dfrac{1}{\sqrt{2}}(2x - 3y) + \dfrac{1}{\sqrt{2}}(-3x + 4y) = \dfrac{1}{\sqrt{2}}(-x + y)$;

$\mathcal{D}_w f(P) = \nabla f \cdot w = (2x - 3y,\ -3x + 4y) \cdot \left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right) =$

$\dfrac{1}{\sqrt{2}}(2x - 3y) - \dfrac{1}{\sqrt{2}}(-3x + 4y) = \dfrac{1}{\sqrt{2}}(5x - 7y)$.

Put $\begin{cases} \frac{1}{\sqrt{2}}(-x + y) = \sqrt{2} \\ \frac{1}{\sqrt{2}}(5x - 7y) = 0 \end{cases} \Rightarrow \begin{cases} -x + y = 2 \\ 5x - 7y = 0 \end{cases} \Rightarrow \begin{cases} x = -7 \\ y = -5 \end{cases}$. Point

$P = (-7,\ -5).$