UNIVERSITÁ DEGLI STUDI DI SIENA
Scuola di Economia e Management
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Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 16/9/2025

I M 1) Calculate the following cubic roots $\sqrt[3]{i^{10}}$.
$i^{10} = (i^2)^5 = (-1)^5 = -1 = cos\,\pi + i\,sin\,\pi$. It follows $\sqrt[3]{i^{10}} = \sqrt[3]{cos\,\pi + i\,sin\,\pi}$ and appling De Moivre's formula:
$\sqrt[3]{cos\,\pi + i\,sin\,\pi} = cos\left(\frac{\pi + 2k\pi}{3}\right) + i\,sin\left(\frac{\pi + 2k\pi}{3}\right)$ with $k = 0, 1, 2$. The three roost are:
$k = 0 \rightarrow z_0 = cos\left(\frac{\pi}{3}\right) + i\,sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i\,;$
$k = 1 \rightarrow z_1 = cos(\pi) + i\,sin(\pi) = -1\,;$
$k = 2 \rightarrow z_2 = cos\left(\frac{5}{3}\pi\right) + i\,sin\left(\frac{5}{3}\pi\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i = \overline{z_0}\,.$

I M 2) Given the matrix $\mathbb{A} = \begin{bmatrix} a & 1 & 1 \\ 1 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ and knowing that $(1, 0, 1)$ is an eigenvector associated to the eigenvalue $\lambda = 1$; find the value of the parameters a and b and calculate the value of the other eigenvalues of the matrix $\mathbb{A}$.
If $(1, 0, 1)$ is an eigenvector associated to the eigenvalue $\lambda = 1$,
$(\mathbb{A} - \mathbb{I}) \cdot (1, 0, 1) = \begin{bmatrix} a-1 & 1 & 1 \\ 1 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 1+b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, easily we get $a = 0$ and $b = -1$, with $\mathbb{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. For calculating the other eigenvalues of the matrix $\mathbb{A}$ we start with the calculus of the characteristic polynomial of the matrix;
$P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda-1 & 1 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda & -1 \\ -1 & \lambda-1 \end{vmatrix} =$
$(\lambda-1)(\lambda(\lambda-1)-1) = (\lambda-1)(\lambda^2-\lambda-1)$. Putting $P_{\mathbb{A}}(\lambda) = 0$ we find the three eigenvalues of matrix $\mathbb{A}$: for $\lambda - 1 = 0$ we have $\lambda_1 = 1$ and for $\lambda^2 - \lambda - 1 = 0$ we get $\lambda_{2,3} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$; $\lambda_2 = \frac{1-\sqrt{5}}{2}$ and $\lambda_3 = \frac{1+\sqrt{5}}{2}$.

I M 3) Given the linear homogeneous system $\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ mx_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + mx_2 + mx_3 + mx_4 = 0 \end{cases}$, find, on varying the parameter m, the dimension of the linear space of its solutions.
To solve the exercise we reduce by elementary operations on the lines of the matrix associated at the linear homogeneous system:

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ m & 1 & 1 & 1 \\ 1 & m & m & m \end{bmatrix} \xrightarrow[R_3 \mapsto R_3 - R_1]{R_2 \mapsto R_2 - m \cdot R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1-m & 1-m & 1-m \\ 0 & m-1 & m-1 & m-1 \end{bmatrix} \xrightarrow{R_3 \mapsto R_3 + R_2}$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1-m & 1-m & 1-m \\ 0 & 0 & 0 & 0 \end{bmatrix}$. From the last matrix we observe that if $m \neq 1$ the rank of the matrix is 2, otherwise is 1 and therefor if we indicate with $\mathcal{S}_m$ the linear space, depending from m, of the solutions of the linear homogeneous system we have $dim(\mathcal{S}_m) = \begin{cases} 2 & m \neq 1 \\ 3 & m = 1 \end{cases}$.

I M 4) Given a linear map $F: \mathbb{R}^3 \to \mathbb{R}^3$, we know that vectors $v_1 = (1,1,0)$ and $v_2 = (0,0,1)$ belong to the Image of F and vector $v_3 = (1,0,0)$ belongs to the Kernel of F. Calculate the dimension of the Image and the dimension of the Kernel, and for the Image find a basis.

Vectors v_1 and v_2 are two linear indipendent vectors that belong to the Image of F and consequently $2 \leq dim(Imm(F)) \leq dim(\mathbb{R}^3) = 3$, thus $2 \leq dim(Imm(F)) \leq 3$; vector v_3 isn't a null vector that belongs to the Kernel of F, consequently $1 \leq dim(Ker(F)) \leq dim(\mathbb{R}^3) = 3$, thus $1 \leq dim(Ker(F)) \leq 3$. By the Rank-Nullity Theorem we know that for linear map F is true that $dim(Imm(F)) + dim(Ker(F)) = dim(\mathbb{R}^3)$ and easily follows $dim(Imm(F)) = 2$ and $dim(Ker(F)) = 1$.

Knowing that $dim(Imm(F)) = 2$ and from the linear indipendency of v_1 and v_2 we can take, as a basis for the image, the set of the two vectors: $\mathcal{B}_{Ima(F)} = \{v_1, v_2\}$.

II M 1) Given the equation $f(x,y) = e^{x^2+y^2} - e^{x+y} = 0$ satisfied at the point $(0,0)$, verify that with it an implicit function $y = y(x)$ can be defined and then calculate, for this implicit function, the first and second derivatives.

$f(0,0) = e^0 - e^0 = 1 - 1 = 0$, condition is satisfied in point $(0,0)$.

$\nabla f = \left(e^{x^2+y^2} \cdot 2x - e^{x+y}, e^{x^2+y^2} \cdot 2y - e^{x+y}\right)$, $\nabla f(0,0) = (-1, -1)$.

At point $(0,0)$ $f'_y(0,0) \neq 0$, thus the proposed condition defines a implicit function $y = y(x)$ with $y'(0) = -\dfrac{f'_x(0,0)}{f'_y(0,0)} = -1$. The second order derivative is

$$y''(0) = -\frac{f''_{xx}(0,0) + 2\,f''_{xy}(0,0) \cdot y'(0) + f''_{yy}(0,0) \cdot (y'(0))^2}{f'_y(0,0)} =$$

$f''_{xx}(0,0) - 2\,f''_{xy}(0,0) + f''_{yy}(0,0)$. The second order partial derivatives of function f are: $f''_{xx} = e^{x^2+y^2} \cdot (2x)^2 + e^{x^2+y^2} \cdot 2 - e^{x+y}$, $f''_{xy} = e^{x^2+y^2} \cdot 2x \cdot 2y - e^{x+y}$ and $f''_{yy} = e^{x^2+y^2} \cdot (2y)^2 + e^{x^2+y^2} \cdot 2 - e^{x+y}$; with calculated on point $(0,0)$ give us: $f''_{xx}(0,0) = f''_{yy}(0,0) = 1$ and $f''_{xy}(0,0) = -1$. The second order derivative is $y''(0) = 1 + 2 + 1 = 4$.

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x + y \\ \text{u.c.: } \begin{cases} x^2 + y^2 \leq 1 \\ y \leq 0 \end{cases} \end{cases}$.

The function f is a polynomial, continuos function, the admissible region is an half disk with center $(0,0)$ and radius 1, a bounded and closed set, therefore f presents absolute maximum and minimum in the admissible region, constraints are qualified at any point of the border of the admissible region. The Lagrangian function is $\mathcal{L}(x,y,\lambda,\mu) = x + y - \lambda(x^2 + y^2 - 1) - \mu(y)$ with

$\nabla\mathcal{L} = (1-2\lambda x, 1-2\lambda y-\mu, -(x^2+y^2-1), -(y))$.

$I^\circ\ CASE\ (free\ optimization)$:

$$\begin{cases}\lambda=\mu=0\\1=0\\1=0\\x^2+y^2\le 1\\y\le 0\end{cases}$$; system impossible.

$II^\circ\ CASE\ (constrained\ optimization\ \text{-}\ first\ constraint\ active)$:

$$\begin{cases}\lambda\neq 0,\mu=0\\1-2\lambda x=0\\1-2\lambda y=0\\x^2+y^2=1\\y\le 0\end{cases}\Rightarrow\begin{cases}\lambda\neq 0,\mu=0\\x=\frac{1}{2\lambda}\\y=\frac{1}{2\lambda}\\x^2+y^2=1\\y\le 0\end{cases}\Rightarrow\begin{cases}\lambda\neq 0,\mu=0\\x=\frac{1}{2\lambda}\\y=\frac{1}{2\lambda}\\\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}=1\\y\le 0\end{cases}\Rightarrow\begin{cases}\lambda\neq 0,\mu=0\\x=\frac{1}{2\lambda}\\y=\frac{1}{2\lambda}\\\lambda^2=\frac{1}{2}\\y\le 0\end{cases}\Rightarrow$$

$$\begin{cases}\lambda\neq 0,\mu=0\\x=\pm\frac{1}{2}\sqrt{2}\\y=\pm\frac{1}{2}\sqrt{2}\\\lambda=\pm\frac{1}{2}\sqrt{2}\\y\le 0\end{cases}$$; point $P_1\left(\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2}\right)$ isn't admissible because $y\nleq 0$ while point $P_2\left(-\frac{1}{2}\sqrt{2},-\frac{1}{2}\sqrt{2}\right)$ is admissible and a candidate to minimum because $\lambda=-\frac{1}{2}\sqrt{2}$ is negative.

$III^\circ\ CASE\ (constrained\ optimization\ \text{-}\ second\ constraint\ active)$:

$$\begin{cases}\lambda=0,\mu\neq 0\\1=0\\1-\mu=0\\x^2+y^2\le 1\\y=0\end{cases}$$; system impossible.

$IIII^\circ\ CASE\ (constrained\ optimization\ \text{-}\ both\ constraints\ active)$:

$$\begin{cases}\lambda\neq 0,\mu\neq 0\\1-2\lambda x=0\\1-2\lambda y-\mu=0\\x^2+y^2=1\\y=0\end{cases}\Rightarrow\begin{cases}\lambda\neq 0,\mu\neq 0\\x=\frac{1}{2\lambda}\\1-\mu=0\\x^2=1\\y=0\end{cases}\Rightarrow\begin{cases}\lambda\neq 0,\mu\neq 0\\\lambda=\pm\frac{1}{2}\\\mu=1\\x=\pm 1\\y=0\end{cases}$$; point $P_3(1,0)$ is a candidate to maximum because the two lagrange multipliers are both positive while point $P_4(-1,0)$ has the two lagrange multipliers with opposite signs, no minimum no maximum. We can conclude that $minf=f(P_2)=-\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}=-\sqrt{2}$ and $Maxf=f(P_3)=1+0=1$.

II M 3) Given the function $f(x,y,z)=x^2+4xy^2+2yz$, analyze the nature of its stationary point.

$\nabla f=\left(2x+4y^2, 8xy+2z, 2y\right)$.

FOC:

$$\begin{cases}2x+4y^2=0\\8xy+2z=0\\2y=0\end{cases}\Rightarrow\begin{cases}2x=0\\2z=0\\y=0\end{cases}\Rightarrow\begin{cases}x=0\\z=0\\y=0\end{cases}$$. A unique critical point $O=(0,0,0)$.

SOC:

$\mathcal{H}_f = \begin{bmatrix} 2 & 8y & 0 \\ 8y & 8x & 2 \\ 0 & 2 & 0 \end{bmatrix}$, $\mathcal{H}_f(O) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$; because a principal minor of order two $\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = -4 < 0$, point O is a saddle point.

II M 4) Given the function $f(x, y) = (x + y)e^{x+y}$ and the unit vector $v = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$; find the two directional derivatives $\mathcal{D}_v f(0, 0)$ and $\mathcal{D}^2_{v,v} f(0, 0)$.

Function f is a differentiable function in the set $\mathbb{R}^2$, the two directional derivatives can be calculated as $\mathcal{D}_v f(0, 0) = \nabla f(0, 0) \cdot v$ and $\mathcal{D}^2_{v,v} f(0, 0) = v^T \cdot \mathcal{H} f(0, 0) \cdot v$. For function f is true that $f'_x = f'_y$ and $f''_{xx} = f''_{xy} = f''_{yy}$;
$f'_x = e^{x+y} + (x + y)e^{x+y} = (1 + x + y)e^{x+y}$, $f'_x(0, 0) = 1$,
$f''_{xx} = e^{x+y} + (1 + x + y)e^{x+y} = (2 + x + y)e^{x+y}$ and $f''_{xx}(0, 0) = 2$; the two directional derivatives are $\mathcal{D}_v f(0, 0) = (1, 1) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$ and

$$\mathcal{D}^2_{v,v} f(0, 0) = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}^T \cdot \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \cdot \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix} = 2 + 2 = 4.$$