UNIVERSITA' DEGLI STUDI DI SIENA
Facoltà di Economia "R. Goodwin"
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Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 18/3/2024

I M 1) Find all the complex numbers z such that their immaginary part are equal 2 and the module of complex number $z+i$ is equal 5. For every complex number z found, calculate its argument.
If the immaginary part of complex number is equal 2, $z = a + 2i$ and $z + i = a + 3i$ with the module $\rho = \sqrt{a^2+9}$. Put $\sqrt{a^2+9} = 5$ follow $a^2 + 9 = 25$ and $a^2 = 16$ with $a = \pm 4$; the request complex numbers are $z_1 = 4 + 2i$ and $z_2 = -4 + 2i$. For their arguments remember that if $a \neq 0$, the argument of z can be calculated as $\theta = arctg\left(\frac{b}{a}\right)$ if a is positive and $\theta = \pi + arctg\left(\frac{b}{a}\right)$ if a is negative, follow that for z_1 and z_2 we get $\theta_1 = arctg\left(\frac{2}{4}\right) = arctg\left(\frac{1}{2}\right)$ and $\theta_2 = \pi + arctg\left(-\frac{2}{4}\right) = \pi - arctg\left(\frac{1}{2}\right)$; $\theta_1 \approx 0.46$ radiants, $\theta_2 \approx 2.68$ radiants.

I M 2) Given the matrix $\mathbb{A} = \begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 1 \\ 2 & k & -2 \end{bmatrix}$ and knowing that -1 is an eigenvalue of the matrix; study if the matrix $\mathbb{A}$ is diagonalizable or not.
At the first step we calculate the characteristic polynomial of matrix $\mathbb{A}$;

$$P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda-2 & -2 & 1 \\ 1 & \lambda-2 & -1 \\ -2 & -k & \lambda+2 \end{vmatrix} = (\lambda-2)\cdot\begin{vmatrix} \lambda-2 & -1 \\ -k & \lambda+2 \end{vmatrix} +$$

$$+2\cdot\begin{vmatrix} 1 & -1 \\ -2 & \lambda+2 \end{vmatrix} + \begin{vmatrix} 1 & \lambda-2 \\ -2 & -k \end{vmatrix} = (\lambda-2)\cdot((\lambda-2)(\lambda+2)-k) +$$

$$+2\cdot(\lambda+2-2) + (-k+2(\lambda-2)) = (\lambda-2)\cdot(\lambda^2-4\lambda-4-k) + 2\lambda +$$

$+(2\lambda-4-k) = \lambda^3 - 6\lambda^2 + (8-k)\lambda + k + 4$. If -1 is an eigenvalue of the matrix, -1 is a root of the characteristic polynomial thus $P_{\mathbb{A}}(-1) = 0$ and $P_{\mathbb{A}}(-1) = 2k - 11$; put $2k - 11 = 0$ easily we find $k = \frac{11}{2}$. Matrix

$$\mathbb{A} = \begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 1 \\ 2 & \frac{11}{2} & -2 \end{bmatrix} \text{ and } P_{\mathbb{A}}(\lambda) = \lambda^3 - 6\lambda^2 + \frac{5}{2}\lambda + \frac{19}{2} =$$

$$(\lambda+1)\cdot\left(\lambda^2 - 7\lambda + \frac{19}{2}\right).$$

Now we calculate the remaining two eigenvalues putting $\lambda^2 - 7\lambda + \frac{19}{2} = 0$, the equation has solutions $\frac{7\pm\sqrt{11}}{2}$ and the three eigenvalues of $\mathbb{A}$ are $\lambda_1 = 1$ and

$\lambda_{2,3} = \dfrac{7 \pm \sqrt{11}}{2}$; matrix is diagonalizable becouse its three eigenvalues are one to one distinct.

I M 3) Given the linear system $\begin{cases} mx_1 + mx_2 + mx_3 = 0 \\ mx_1 + mx_2 + x_3 = 0 \\ mx_1 + x_2 + x_3 = 0 \end{cases}$, where m is a real parameter. We indicate with S_m the set of its solutions, study, varing m, the dimention of the set S_m, and when the dimention is bigger, find a basis for S_m.

The matrix associated to the system is $\begin{bmatrix} m & m & m \\ m & m & 1 \\ m & 1 & 1 \end{bmatrix}$. We reduce the matrix by elementary operations on its lines:

$$\begin{bmatrix} m & m & m \\ m & m & 1 \\ m & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \mapsto R_3 - R_1]{R_2 \mapsto R_2 - R_1} \begin{bmatrix} m & m & m \\ 0 & 0 & 1-m \\ 0 & 1-m & 1-m \end{bmatrix} \xrightarrow{R_2 \circlearrowleft R_3} \begin{bmatrix} m & m & m \\ 0 & 1-m & 1-m \\ 0 & 0 & 1-m \end{bmatrix}.$$

The determinant of the reduced matrix is $\begin{vmatrix} m & m & m \\ 0 & 1-m & 1-m \\ 0 & 0 & 1-m \end{vmatrix} = m(1-m)^2$ and it is different from zero if and only if $m \neq 0$ and $m \neq 1$, rank of matrix $\begin{bmatrix} m & m & m \\ m & m & 1 \\ m & 1 & 1 \end{bmatrix}$ is three if and only if $m \neq 0$ and $m \neq 1$ and this imply that in this case the dimention of S_m is zero. If $m = 1$ the matrix associated to the system is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, with rank equal at 1 and dimention of S_m is 2; finally if $m = 0$ the matrix associated to the system is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, with rank equal at 2 and dimention of S_m is 1. In conclusion $dim(S_m) = \begin{cases} 2 & \text{if } m = 1 \\ 1 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$. Dimention of S_m is bigger if $m = 1$, and in this case the system is reduced to the unique equazion $x_1 + x_2 + x_3 = 0$ or $x_3 = -x_1 - x_2$, and a generic element of S_1 is $(x_1, x_2, -x_1 - x_2) = x_1(1, 0, -1) + x_2(0, 1, -1)$, a basis for S_1 is the set of vectors $\mathcal{B}_{S_1} = \{(1, 0, -1), (0, 1, -1)\}$. To be thorough we consider also the case $m = 0$, in this situation the system is reduced to $\begin{cases} x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = 0 \\ x_2 = 0 \end{cases}$, and a generic element of S_0 is $(x_1, 0, 0) = x_1(1, 0, 0)$, a basis for S_0 is the set $\mathcal{B}_{S_0} = \{(1, 0, 0)\}$.

I M 4) Given a linear map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$, we know that:

1. $F(1, 1, 1) = (0, 0, 0)$;
2. $F(0, 0, 1) = (0, 0, 1)$;
3. $F(1, 0, 0) = (1, 0, 0)$.

Find the dimention of its image and the dimention of its kernel; and for both, image and kernel, set a basis.

For the linear map F we know that the vector $(1, 1, 1)$ belongs in the kernel of F, thus the dimention of the kernel is at least one: $dim(KerF) \geq 1$. The two linear indipendent vectors $(0, 0, 1)$ and $(1, 0, 0)$ belong in the image of F, thus the dimention of the image is at least two: $dim(ImaF) \geq 2$. By the dimention Theorem is known that for the linear map F, $dim(KerF) + dim(ImaF) = dim(\mathbb{R}^3) = 3$, an by the two previous inequalities easily we conclude that $dim(KerF) = 1$ and $dim(ImaF) = 2$. The two basis for the spaces, kernel and image, can be easily found as $\mathcal{B}_{KerF} = \{(1,1,1)\}$ and $\mathcal{B}_{ImaF} = \{(0,0,1),(1,0,0)\}$.

II M 1) The equation $f(x,y) = 0$ satisfied on point $P(x_P, y_P)$, defined an implicit function $y = y(x)$. We know that the gradient of f on point P is $\nabla f(P) = (1, -1)$ and the Hessian matrix of f on point P is $\mathcal{H}f(P) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. For this implicit function calculate the first and second derivatives $y'(x_P)$ and $y''(x_P)$.

By the Dini's Theorem if $f'_y(P) \neq 0$, $y'(x_P) = -\dfrac{f'_x(P)}{f'_y(P)} = -\dfrac{1}{-1} = 1$; and

$$y''(x_P) = -\frac{f''_{x,x}(P) + 2 \cdot f''_{x,y}(P) \cdot y'(x_P) + f''_{y,y}(P) \cdot (y'(x_P))^2}{f'_y(P)} =$$

$$-\frac{f''_{x,x}(P) + 2 \cdot f''_{x,y}(P) + f''_{y,y}(P)}{f'_y(P)} =$$

$$-\frac{1 + 2 \cdot (-1) + 1}{-1} = 0.$$

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x^3 + y^3 \\ \text{u.c.: } x^2 + y^2 \leq 8 \end{cases}$.

The function f is a polynomial, continuos function, the admissible region is the interior region of a circunference, a bounded and closed set; constraint is qualified on circunference, therefore f presents absolute maximum and minimum in the admissible region. The Lagrangian function is

$\mathcal{L}(x,y,\lambda) = x^3 + y^3 - \lambda(x^2 + y^2 - 8)$ with

$\nabla\mathcal{L} = (3x^2 - 2\lambda x, 3y^2 - 2\lambda y, -(x^2 + y^2 - 8))$.

$I^\circ\ CASE$ ($free\ optimization$):

$\begin{cases} \lambda = 0 \\ 3x^2 = 0 \\ 3y^2 = 0 \\ x^2 + y^2 - 8 \leq 0 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ x = 0 \\ y = 0 \\ -8 \leq 0 \end{cases}$; point $(0,0)$ is admissible, $\mathcal{H}f = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$ and

$\mathcal{H}f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathcal{H}_2(0,0) = 0$. We haven't any information about the nature of point $(0,0)$.

$II^\circ\ CASE$ ($constrained\ optimization$):

$\begin{cases} \lambda \neq 0 \\ 3x^2 - 2\lambda x = 0 \\ 3y^2 - 2\lambda y = 0 \\ x^2 + y^2 - 8 = 0 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ x(3x - 2\lambda) = 0 \\ y(3y - 2\lambda) = 0 \\ x^2 + y^2 = 8 \end{cases}$; we must evaluate four possibilities:

a: if $x = 0$ and $y = 0$, $0^2 + 0^2 \neq 8$; point $(0,0)$ isn't admissible;

b: if $x = 0$ and $y = \frac{2}{3}\lambda$, we get $\frac{4}{9}\lambda^2 = 8 \Rightarrow \lambda^2 = 18 \Rightarrow \lambda = \pm 3\sqrt{2}$, with $y = \pm 2\sqrt{2}$; point $P_1\left(0, 2\sqrt{2}\right)$ is a candidate to maximum ($\lambda > 0$), while point $P_2\left(0, -2\sqrt{2}\right)$ is a candidate to minimum ($\lambda < 0$);

c: if $y = 0$ and $x = \frac{2}{3}\lambda$, we get again $\frac{4}{9}\lambda^2 = 8 \Rightarrow \lambda = \pm 3\sqrt{2}$, with $x = \pm 2\sqrt{2}$; point $P_3\left(2\sqrt{2}, 0\right)$ is a candidate to maximum ($\lambda > 0$), while point $P_4\left(-2\sqrt{2}, 0\right)$ is a candidate to minimum ($\lambda < 0$);

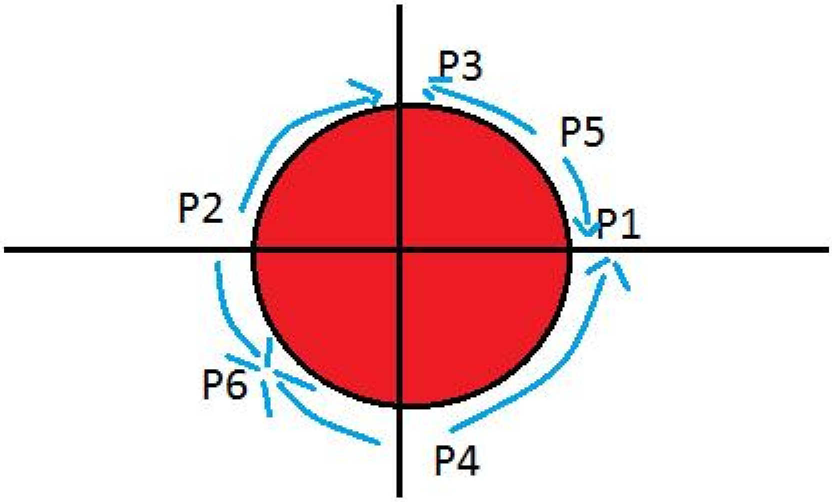
d: if $x = \dfrac{2}{3}\lambda$ and $y = \dfrac{2}{3}\lambda$, we get $\dfrac{8}{9}\lambda^2 = 8 \Rightarrow \lambda^2 = 9 \Rightarrow \lambda = \pm 3$, with $x = \pm 2$ and $y = \pm 2$; point $P_5(2,2)$is a candidate to maximum ($\lambda > 0$), while point $P_6(-2,-2)$ is a candidate to minimum ($\lambda < 0$). $f(P_{1,3}) = 16\sqrt{2}$, $f(P_5) = 16 < 16\sqrt{2}$, f presents absolute maximum equal $16\sqrt{2}$ in points $\left(0, 2\sqrt{2}\right)$ and $\left(2\sqrt{2}, 0\right)$; $f(P_{2,4}) = -16\sqrt{2}$, $f(P_6) = -16 > -16\sqrt{2}$, f presents absolute minimum equal $-16\sqrt{2}$ in points $\left(0, -2\sqrt{2}\right)$ and $\left(-2\sqrt{2}, 0\right)$.

To analize the nature of point P_5 and P_6 we study the function f along the upper and the lower border of the admissible region. Rewrite the circunference's equation as $y^2 = 8 - x^2$, the upper and the lower borders of the admissible region are respectively $y = +\sqrt{8 - x^2}$ and $y = -\sqrt{8 - x^2}$.

In the upper border consider the function $f\left(x, +\sqrt{8 - x^2}\right) = x^3 + \left(\sqrt{8 - x^2}\right)^3 = g(x)$, $g'(x) = 3x^2 + 3\left(\sqrt{8 - x^2}\right)^2 \cdot \dfrac{-2x}{2\sqrt{8 - x^2}} = 3x\left(x - \sqrt{8 - x^2}\right)$; $g'(x) > 0$ if and only if $-2\sqrt{2} < x < 0$ or $2 < x < 2\sqrt{2}$. Along the upper border, function f is increasing for $-2\sqrt{2} < x < 0$ and $2 < x < 2\sqrt{2}$, decreasing for $0 < x < 2$, P_5 is a false maximum (minimum point along the border).

By the exchangeability on variables in function f ($f(x,y) = f(y,x)$), similar results can be achaived in the lower border.

In the graphic below, the admissible region, in red, and the behaviour of f along the border rappresented by the turquoise arrows.



II M 3) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x^2 + y^2 \\ \text{u.c.: } x - y = 4 \end{cases}$.

The Lagrangian function of the problem is $\mathcal{L}(x,y,\lambda) = x^2 + y^2 - \lambda(x - y - 4)$ with $\nabla\mathcal{L} = (2x - \lambda, 2y + \lambda, -(x - y - 4))$.

FOC:

$$\begin{cases} 2x-\lambda=0 \\ 2y+\lambda=0 \\ x-y=4 \end{cases} \Rightarrow \begin{cases} x=\lambda/2 \\ y=-\lambda/2 \\ \lambda/2+\lambda/2=4 \end{cases} \Rightarrow \begin{cases} x=\lambda/2 \\ y=-\lambda/2 \\ \lambda=4 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=-2 \\ \lambda=4 \end{cases}$$; one constraint critical points $P=(2,-2)$.

SOC:

$\overline{\mathcal{H}} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, with $|\overline{\mathcal{H}}| = \begin{vmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} =$
$-2-2=-4$. $|\overline{\mathcal{H}}(P)|<0$, P point of minimum with $f(P)=8$.

II M 4) Given the function $f(x,y)=e^{x+y}-e^{x-y}$ and the unit vector $v=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$;
calculate on point $(0,0)$ the directional derivatives $\mathcal{D}_v f(0,0)$ and $\mathcal{D}^{(2)}_{v,v} f(0,0)$.

$\nabla f(x,y)=(e^{x+y}-e^{x-y},e^{x+y}+e^{x-y})$, $\nabla f(0,0)=(0,2)$,

$\mathcal{D}_v f(0,0)=\nabla f(0,0)\cdot v=(0,2)\cdot\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)=\sqrt{3}$.

$\mathcal{H}f(x,y)=\begin{bmatrix} e^{x+y}-e^{x-y} & e^{x+y}+e^{x-y} \\ e^{x+y}+e^{x-y} & e^{x+y}-e^{x-y} \end{bmatrix}$ and $\mathcal{H}f(0,0)=\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ with

$\mathcal{D}^{(2)}_{v,v} f(0,0)=v^T\cdot\mathcal{H}f(0,0)\cdot v=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)\cdot\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\cdot\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}=$
$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)\cdot\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}=\sqrt{3}$.