UNIVERSITÁ DEGLI STUDI DI SIENA
Scuola di Economia e Management
A.A. 2025/26
Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 12/1/2026

I M 1) Given the complex number $z = i^{12} + i^{13}$; calculate its square roots.
Remember that $i^4 = 1$, thus $z = i^{12} + i^{13} = i^{12}(1+i) = (i^4)^3(1+i) = 1^3(1+i) = 1 + i = \sqrt{2}\left(cos\frac{\pi}{4} + i\,sin\frac{\pi}{4}\right)$. For the roots of z we apply the classical formula:

$$\sqrt{z} = z_k = \sqrt[4]{2}\left(cos\left(\frac{\pi/4 + 2k\pi}{2}\right) + i\,sin\left(\frac{\pi/4 + 2k\pi}{2}\right)\right) \;\; k = 0, 1$$
$$= \sqrt[4]{2}\left(cos\left(\frac{\pi}{8} + k\pi\right) + i\,sin\left(\frac{\pi}{8} + k\pi\right)\right) \;\; k = 0, 1.$$

The two roots are:

$$k = 0 \rightarrow \; z_0 = \sqrt[4]{2}\left(cos\frac{\pi}{8} + i\,sin\frac{\pi}{8}\right) = \sqrt{\frac{\sqrt{2}+1}{2}} + i\sqrt{\frac{\sqrt{2}-1}{2}}\,;$$
$$k = 1 \rightarrow \; z_1 = \sqrt[4]{2}\left(cos\frac{9\pi}{8} + i\,sin\frac{9\pi}{8}\right) = -\sqrt{\frac{\sqrt{2}+1}{2}} - i\sqrt{\frac{\sqrt{2}-1}{2}} = -z_0\,.$$

I M 2) Consider the matrix $\mathbb{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$. Calculate its eigenvalues and study if the matrix $\mathbb{A}$ is a diagonalizable one.

The first step is the calculus of the characteristic polynomial of the matrix:

$$P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda-1 & 0 & 0 & -1 \\ 0 & \lambda-1 & -1 & 0 \\ 0 & 1 & \lambda+1 & 0 \\ 1 & 0 & 0 & \lambda+1 \end{vmatrix} =$$
$$(\lambda-1)\begin{vmatrix} \lambda-1 & -1 & 0 \\ 1 & \lambda+1 & 0 \\ 0 & 0 & \lambda+1 \end{vmatrix} + \begin{vmatrix} 0 & \lambda-1 & -1 \\ 0 & 1 & \lambda+1 \\ 1 & 0 & 0 \end{vmatrix} =$$
$$(\lambda-1)(\lambda+1)\begin{vmatrix} \lambda-1 & -1 \\ 1 & \lambda+1 \end{vmatrix} + \begin{vmatrix} \lambda-1 & -1 \\ 1 & \lambda+1 \end{vmatrix} = (\lambda^2 - 1 + 1)\begin{vmatrix} \lambda-1 & -1 \\ 1 & \lambda+1 \end{vmatrix} =$$

$\lambda^2((\lambda-1)(\lambda+1)+1) = \lambda^4$. Putting $P_{\mathbb{A}}(\lambda) = 0$ we find the unique eigenvalue of matrix $\mathbb{A}$: $\lambda = 0$ with algebraic multiplicity equal four. To verify if the matrix is diagonalizable, we must find the geometric multiplicity of the unique eigenvalue, for this goal we calculate the rank of matrix

$0 \cdot \mathbb{I} - \mathbb{A} = -\mathbb{A} = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, it's easy note that from matrix $\mathbb{A}$ we

can define a principal minor of order 2, $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ with determinant different from 0 and matrix $-\mathbb{A}$ has the third raw equal to the opposite of the second raw and the fourth raw equal to the opposite of the first raw, thus $Rank(\mathbb{A})=2$ and the geometric multiplicity of eigenvalue 0 is two. The matrix $\mathbb{A}$ isn't diagonalizable.
Alternative solution: if $\boldsymbol{v}=(v_1,v_2,v_3,v_4)$ is an eigenvector associated to the unique eigenvalue $\lambda=0$, $\boldsymbol{v}$ must satisfies the equation
$\mathbb{A}\boldsymbol{v}=0\boldsymbol{v}=\mathbf{0}\Rightarrow\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & -1 & -1 & 0\\ -1 & 0 & 0 & -1\end{bmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\\ v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ that implies
$v_1+v_4=0$ and $v_2+v_3=0$; from the last two conditions we get that an eigenvector associated to the unique eigenvalue $\lambda=0$ is
$(v_1,v_2,-v_2,-v_1)=v_1(1,0,0,-1)+v_2(0,1,-1,0)$. The eigenspace associated to $\lambda=0$ is generated by the two vectors $(1,0,0,-1)$ and $(0,1,-1,0)$, thus the geometric multiplicity of eigenvalue 0 is two. The matrix $\mathbb{A}$ isn't diagonalizable.
A second alternative solution: matrix $-\mathbb{A}\neq\mathbb{O}$ (a null matrix), this implies that $Rank(-\mathbb{A})\geq 1$ and the geometric multiplicity of the eigenvalue $\lambda=0$ is $m^g_{\lambda=0}=4-Rank(-\mathbb{A})\leq 3<4=m^a_{\lambda=0}$. The matrix $\mathbb{A}$ isn't diagonalizable.
I M 3) Given a linear map $F:\mathbb{R}^3\to\mathbb{R}^2$, we know that:

1. vector $(1,0,-1)$ belongs to the kernel of F;
2. $F(1,1,1)=(1,0)$ and $F(1,-1,1)=(0,-1)$.

Find the matrix $\mathbb{A}_F$ associated with the linear map and calculate the dimention of the kernel and the dimention of the image of the linear map.
$F:\mathbb{R}^3\to\mathbb{R}^2$ and matrix $\mathbb{A}_F$ associated to F is a 2×3 matrix, $\mathbb{A}_F=\begin{bmatrix}\alpha & \beta & \chi\\ \delta & F & \epsilon\end{bmatrix}$. If vector $(1,0,-1)$ belongs to the kernel of F,
$\begin{bmatrix}\alpha & \beta & \chi\\ \delta & F & \epsilon\end{bmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix}=\begin{pmatrix}\alpha-\chi\\ \delta-\epsilon\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ and easily follow $\mathbb{A}_F=\begin{bmatrix}\alpha & \beta & \alpha\\ \delta & F & \delta\end{bmatrix}$.
From $F(1,1,1)=(1,0)$ we get $\begin{bmatrix}\alpha & \beta & \alpha\\ \delta & F & \delta\end{bmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}2\alpha+\beta\\ 2\delta+F\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$ with
$\beta=1-2\alpha$ and $F=-2\delta$, $\mathbb{A}_F=\begin{bmatrix}\alpha & 1-2\alpha & \alpha\\ \delta & -2\delta & \delta\end{bmatrix}$. Finally if
$F(1,-1,1)=(0,-1)$, $\begin{bmatrix}\alpha & 1-2\alpha & \alpha\\ \delta & -2\delta & \delta\end{bmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}4\alpha-1\\ 4\delta\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix}\Rightarrow$
$\alpha=\dfrac{1}{4}$ and $\delta=-\dfrac{1}{4}$; $\mathbb{A}_F=\begin{bmatrix}\frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4}\end{bmatrix}$. Remember that the dimention of the image of a linear map is equal to the rank of the associated matrix and for the map F, the matrix $\mathbb{A}_F$ has a minor of order 2, $\begin{bmatrix}\frac{1}{4} & \frac{1}{2}\\ -\frac{1}{4} & \frac{1}{2}\end{bmatrix}$ with determinant different from 0, matrix $\mathbb{A}_F$ has rank equal two and the dimention of the image of map F is 2; from the Rank-Nullity Theorem $dim(Ima(F))+dim(Ker(F))=dim(\mathbb{R}^3)=3$, so the dimention of the kernel is 1.
Alternative solution: from 1. vector $(1,0,-1)$ belongs to the kernel of F and

2. $F(1,1,1)=(1,0)$ and $F(1,-1,1)=(0,-1)$; we have that $dim(Ker(F))\geq 1$ and $dim(Ima(F))\geq 2$ (because vectors $(1,0)$ and $(0,-1)$ are linear indipendent); from Rank-Nullity Theorem $dim(Ima(F))+dim(Ker(F))=dim(\mathbb{R}^3)=3$, easily we conclude that $dim(Ima(F))=2$ and $dim(Ker(F))=1$.

I M 4) Given the two matrices $\mathbb{A}=\begin{bmatrix}1&0&0\\3&1&0\\2&3&1\end{bmatrix}$ and $\mathbb{B}=\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}$. Calculate the inverse matrix of $\mathbb{A}^T+\mathbb{B}$.

$$\mathbb{A}^T+\mathbb{B}=\begin{bmatrix}1&0&0\\3&1&0\\2&3&1\end{bmatrix}^T+\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}=\begin{bmatrix}1&3&2\\0&1&3\\0&0&1\end{bmatrix}+\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}=$$

$\begin{bmatrix}2&5&5\\0&2&5\\0&0&2\end{bmatrix}=\mathbb{C}$. The inverse matrix of $\mathbb{C}$ is $\mathbb{C}^{-1}=\dfrac{1}{|\mathbb{C}|}(Adj(\mathbb{C}))^T$, where $Adj(\mathbb{C})$ is the adjoint matrix of $\mathbb{C}$. $|\mathbb{C}|=\begin{vmatrix}2&5&5\\0&2&5\\0&0&2\end{vmatrix}=8$.

$$Adj(\mathbb{C})=\begin{bmatrix}\begin{vmatrix}2&5\\0&2\end{vmatrix}&-\begin{vmatrix}0&5\\0&2\end{vmatrix}&\begin{vmatrix}0&2\\0&0\end{vmatrix}\\-\begin{vmatrix}5&5\\0&2\end{vmatrix}&\begin{vmatrix}2&5\\0&2\end{vmatrix}&-\begin{vmatrix}2&5\\0&0\end{vmatrix}\\\begin{vmatrix}5&5\\2&5\end{vmatrix}&-\begin{vmatrix}2&5\\0&5\end{vmatrix}&\begin{vmatrix}2&5\\0&2\end{vmatrix}\end{bmatrix}=\begin{bmatrix}4&0&0\\-10&4&0\\15&-10&4\end{bmatrix}.$$

$$\mathbb{C}^{-1}=\frac{1}{8}\begin{bmatrix}4&0&0\\-10&4&0\\15&-10&4\end{bmatrix}^T=\frac{1}{8}\begin{bmatrix}4&-10&15\\0&4&-10\\0&0&4\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&-\frac{5}{4}&\frac{15}{8}\\0&\frac{1}{2}&-\frac{5}{4}\\0&0&\frac{1}{2}\end{bmatrix}.$$

II M 1) Given the equation $f(x,y,z)=e^{x+y+z}+xyz+z=1$ satisfied at the point $(0,0,0)$, verify that with it an implicit function $z=z(x,y)$ can be defined and then calculate, for this implicit function its gradient vector ∇z.

$f(0,0,0)=e^0+0+0=1$, $f'_x=e^{x+y+z}+yz$, $f'_y=e^{x+y+z}+xz$ and $f'_z=e^{x+y+z}+xy+1$, with $f'_x(0,0,0)=1$, $f'_y(0,0,0)=1$ and $f'_z(0,0,0)=2$.

Since $f'_z(0,0,0)\neq 0$, the equation $f(x,y)=e^{x+y+z}+xyz+z=1$ defines a function $z=z(x,y)$ with $\nabla z=\left(z'_x(0,0),z'_y(0,0)\right)=\left(-\dfrac{f'_x(0,0,0)}{f'_z(0,0,0)},-\dfrac{f'_y(0,0,0)}{f'_z(0,0,0)}\right)=$
$\left(-\dfrac{1}{2},-\dfrac{1}{2}\right)$.

II M 2) Solve the problem $\begin{cases}\text{Max/min } f(x,y)=x-y^2\\ \text{u.c.: } x^2+y^2\leq 4\end{cases}$.

The function f is a polynomial, continuos function, the admissible region is a disk with center $(0,0)$ and radius 2, a bounded and closed set, therefore f presents absolute maximum and minimum in the admissible region, constraint is qualified in any point in the circumference $x^2+y^2=4$. The Lagrangian function is
$\mathcal{L}(x,y,\lambda)=x-y^2-\lambda(x^2+y^2-4)$ with
$\nabla\mathcal{L}=(1-2\lambda x,-2y-2\lambda y,-(x^2+y^2-4))$.

$I^\circ\ CASE$ ($free\ optimization$):

$$\begin{cases} \lambda = 0 \\ 1 = 0 \\ -2y = 0 \\ x^2 + y^2 \leq 4 \end{cases}$$; system impossible.

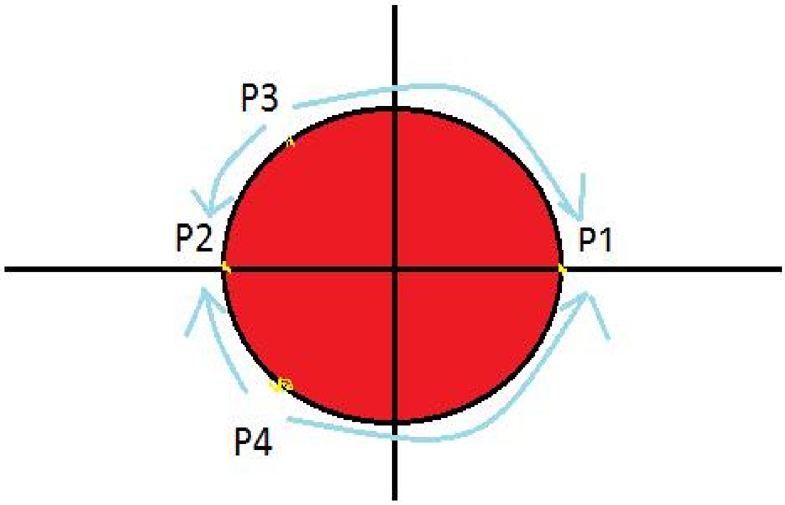
$II^\circ\ CASE$ ($constrained\ optimization$):

$$\begin{cases} \lambda \neq 0 \\ 1 - 2\lambda x = 0 \\ -2y - 2\lambda y = 0 \\ x^2 + y^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ 1 - 2\lambda x = 0 \\ -2y(1+\lambda) = 0 \\ x^2 + y^2 = 4 \end{cases}$$; if $y = 0$, $x = \pm 2$ and $\lambda = \pm \frac{1}{4}$,

otherwise if $\lambda = -1$, $x = -\frac{1}{2}$ and $y = \pm \frac{1}{2}\sqrt{15}$. Four critical points $P_{1,2} = (\pm 2, 0)$, P_1 candidate for maximum ($\lambda > 0$), P_2 candidate for minimum ($\lambda < 0$), and $P_{3,4} = \left(-\frac{1}{2}, \pm \frac{1}{2}\sqrt{15}\right)$, both candidate for minimum ($\lambda < 0$),.

$f(P_{1,2}) = \pm 2$, $f(P_{3,4}) = -\frac{17}{4} < -2$, f presents absolute maximum equal 2 on points $(2,0)$ and absolute minimum equal $-\frac{17}{4}$ on points $\left(-\frac{1}{2}, \pm \frac{1}{2}\sqrt{15}\right)$.

To study the nature of point P_2 we analize the function f along the border $x^2 + y^2 = 4$ with $y^2 = 4 - x^2$; $f(x,y) = x - (4 - x^2) = x^2 + x - 4 = g(x)$; $g'(x) = 2x + 1$ and $g'(x) \geq 0$ iff $x \geq -\frac{1}{2}$, $g(x)$ is increasing for $-\frac{1}{2} \leq x \leq 2$, decreasing for $-2 \leq x \leq -\frac{1}{2}$; point P_2 is a false minimum. In the following the increasing behavior of function f along the border, turquoise arrows, in red the admissible region.



II M 3) Find the minimum of the function $f(x,y,z) = x^2 + y^2 + z^4 + 6z^2$.

$\nabla f = \left(2x, 2y, 4z^3 + 12z\right)$.

FOC:

$$\begin{cases} 2x = 0 \\ 2y = 0 \\ 4z^3 + 12z = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 4z(z^2+3) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$. One critical point $O = (0,0,0)$.

SOC:

$\mathcal{H}_f = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 12z^2+12 \end{bmatrix}$, with $\mathcal{H}_f(O) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 12 \end{bmatrix}$ and $\mathcal{H}_f^1 = 2 > 0$,

$\mathcal{H}_f^2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$, $\mathcal{H}_f^3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 12 \end{vmatrix} = 48 > 0$. $O = (0,0,0)$ is a point of minimum and the minimum of the function is $f(0,0,0) = 0$.

II M 4) Consider the function $f(x,y) = ye^{x+y}$ and the two unit vectors $v_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $v_2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$; at point (x_0, y_0) the two directional derivatives $\mathcal{D}_{v_1}f(x_0,y_0)$ and $\mathcal{D}_{v_2}f(x_0,y_0)$ are respectively equal to -1 and 1. Find the point (x_0, y_0) and calculate the second order directional derivative $\mathcal{D}^{(2)}_{v_1,v_2}f(x_0,y_0)$.

Function f is differentiable in all points,

$\nabla f(x,y) = (ye^{x+y}, e^{x+y} + ye^{x+y}) = (ye^{x+y}, (1+y)e^{x+y})$;

$$\mathcal{D}_{v_1}f(x_0,y_0) = \nabla f \cdot v_1 = (ye^{x+y}, (1+y)e^{x+y}) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(1+2y)e^{x+y};$$

$$\mathcal{D}_{v_2}f(x_0,y_0) = \nabla f \cdot v_2 = (ye^{x+y}, (1+y)e^{x+y}) \cdot \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}e^{x+y};$$

now we solve the system $\begin{cases} \mathcal{D}_{v_1}f(x_0,y_0) = \frac{\sqrt{2}}{2}(1+2y)e^{x+y} = -1 \\ \mathcal{D}_{v_2}f(x_0,y_0) = \frac{\sqrt{2}}{2}e^{x+y} = 1 \end{cases} \Rightarrow$

$\begin{cases} 1+2y = -1 \\ \frac{\sqrt{2}}{2}e^{x+y} = 1 \end{cases} \Rightarrow \begin{cases} y = -1 \\ x = 1 + log\sqrt{2} \end{cases}$. Point $(x_0, y_0) = (1 + log\sqrt{2}, -1)$.

The second order directional derivative is calculated by the product $v_1^T \cdot \mathcal{H}_f(x_0,y_0) \cdot v_2$;

$\mathcal{H}_f = \begin{bmatrix} ye^{x+y} & (1+y)e^{x+y} \\ (1+y)e^{x+y} & (2+y)e^{x+y} \end{bmatrix}$, $\mathcal{H}_f(x_0,y_0) = \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$,

$$\mathcal{D}^{(2)}_{v_1,v_2}f(x_0,y_0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \cdot \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} =$$

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sqrt{2}.$$