UNIVERSITÁ DEGLI STUDI DI SIENA
Scuola di Economia e Management
A.A. 2025/26
Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 3/2/2026

I M 1) Find the solutions of the equation $z^3 - 4z^2 + 9z - 36 = 0$, and calculate the square roots of the unique solution having positive immaginary part.
$z^3 - 4z^2 + 9z - 36 = z^2(z-4) + 9(z-4) = (z-4)(z^2+9)$. Put $(z-4)(z^2+9) = 0$, if $z - 4 = 0$ we get the first solution $z_1 = 4$, while if $z^2 + 9 = 0 \Leftrightarrow z^2 = -9$ we get the other two solutions $z_{2,3} = \pm\sqrt{-9} = \pm\sqrt{9}\cdot\sqrt{-1} = \pm 3i$. The unique solution with positive immaginary part is $z_2 = 3i = 3\left(cos\frac{\pi}{2} + i\,sin\frac{\pi}{2}\right)$ and its square roots can be calculated by De Moivre's Formula: $\sqrt{z_2} = \sqrt{3\left(cos\frac{\pi}{2} + i\,sin\frac{\pi}{2}\right)} =$

$$\sqrt{3}\left(cos\left(\frac{\pi/2 + 2k\pi}{2}\right) + i\,sin\left(\frac{\pi/2 + 2k\pi}{2}\right)\right) =$$

$$\sqrt{3}\left(cos\left(\frac{\pi}{4} + k\pi\right) + i\,sin\left(\frac{\pi}{4} + k\pi\right)\right)\ k = 0, 1.$$

The two roots are:

$$k = 0 \rightarrow\ z_{20} = \sqrt{3}\left(cos\frac{\pi}{4} + i\,sin\frac{\pi}{4}\right) = \frac{\sqrt{6}}{2} + i\frac{\sqrt{6}}{2}\,;$$

$$k = 1 \rightarrow\ z_{21} = \sqrt{3}\left(cos\frac{5\pi}{4} + i\,sin\frac{5\pi}{4}\right) = -\frac{\sqrt{6}}{2} - i\frac{\sqrt{6}}{2} = -z_{20}\,.$$

I M 2) Given the linear map $F: \mathbb{R}^4 \rightarrow \mathbb{R}^2$, with $F(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3 - x_4, mx_1 - mx_2 + x_3 - x_4)$. Find, varing the real parameter m, the dimention of the image of the linear map F, and when this dimention is minimum, find a basis for the kernel and a basis for the image of F.
The matrix $\mathbb{A}_F$ associated at the linear map F is $\begin{bmatrix} 1 & -1 & 1 & -1 \\ m & -m & 1 & -1 \end{bmatrix}$; the dimention of map F is equal at the rank of matrix $\mathbb{A}_F$ and to calculate it we reduce the matrix by elementary operations on its lines:
$\begin{bmatrix} 1 & -1 & 1 & -1 \\ m & -m & 1 & -1 \end{bmatrix} \begin{matrix} C_2 \mapsto C_1 + C_2 \\ C_4 \mapsto C_3 + C_4 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ m & 0 & 1 & 0 \end{bmatrix} C_2 \circlearrowleft C_3 \begin{bmatrix} 1 & 1 & 0 & 0 \\ m & 1 & 0 & 0 \end{bmatrix}$. The last matrix has a submatrix of order 2 with determinant $\begin{vmatrix} 1 & 1 \\ m & 1 \end{vmatrix} = 1 - m$, it is different from 0 if and only if $m \neq 1$, if $m = 1$ $\mathbb{A}_F$ has $R_1 = R_2$ and its rank is 1, we conclude that $dim(IM_F) = Rank(\mathbb{A}_F) = \begin{cases} 2 & \text{if } m \neq 1 \\ 1 & \text{if } m = 1 \end{cases}$, and the dimention is smallest if $m = 1$. In this case a generic element of the image is $(x_1 - x_2 + x_3 - x_4, x_1 - x_2 + x_3 - x_4) = (y, y) = y(1, 1)$, a basis for the image is the set $\mathcal{B}_{IM_F} = \{(1, 1)\}$; for the basis of the kernel we note that a generic element of the linear space (x_1, x_2, x_3, x_4) must satisfied $x_1 - x_2 + x_3 - x_4 = 0$ or $x_4 = x_1 - x_2 + x_3$; a generic element of the kernel is
$(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_1 - x_2 + x_3) =$

$x_1(1,0,0,1)+x_2(0,1,0,-1)+x_3(0,0,1,1)$. A basis for the kernel is the set $\mathcal{B}_{KER_F}=\{(1,0,0,1),(0,1,0,-1),(0,0,1,1)\}$.

I M 3) Given the matrix $\mathbb{A}=\begin{bmatrix}1 & 0 & -2\\ 0 & 0 & 0\\ -2 & 0 & 4\end{bmatrix}$. Calculate its eigenvalues and for the unique eigenvalue having algebraic multiplicity equal one, find a basis for its associated eigenspace.

The first step is the calculus of the characteristic polynomial of the matrix:

$P_{\mathbb{A}}(\lambda)=|\lambda\mathbb{I}-\mathbb{A}|=\begin{vmatrix}\lambda-1 & 0 & 2\\ 0 & \lambda & 0\\ 2 & 0 & \lambda-4\end{vmatrix}=\lambda\begin{vmatrix}\lambda-1 & 2\\ 2 & \lambda-4\end{vmatrix}=$ $\lambda((\lambda-1)(\lambda-4)-4)=\lambda(\lambda^2-5\lambda+4-4)=\lambda^2(\lambda-5)$. Put $P_{\mathbb{A}}(\lambda)=0$ we find an eigenvalue $\lambda=0$ with algebraic multiplicity equal two and the eigenvalue $\lambda=5$ with algebraic multiplicity equal one. To find a basis for the eigenspace associated at the eigenvalue $\lambda=5$, we consider a generic element of the eigenspace $\boldsymbol{v}=(v_1,v_2,v_3)$ that must satisfied $(5\mathbb{I}-\mathbb{A})\boldsymbol{v}=\mathbb{O}$. $(5\mathbb{I}-\mathbb{A})\boldsymbol{v}=\begin{bmatrix}4 & 0 & 2\\ 0 & 5 & 0\\ 2 & 0 & 1\end{bmatrix}\cdot\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}4v_1+2v_3\\ 5v_2\\ 2v_1+v_3\end{pmatrix}$ and by condition $\begin{pmatrix}4v_1+2v_3\\ 5v_2\\ 2v_1+v_3\end{pmatrix}=\mathbb{O}$ we have $v_2=0$ and $v_3=-2v_1$; a generic element of the eigenspace associated at the eigenvalue $\lambda=5$ is the vector $(v_1,0,-2v_1)=v_1(1,0,-2)$. A basis for the eigenspace is the set $\mathcal{B}_{EIS_{\lambda=5}}=\{(1,0,-2)\}$.

I M 4) Study, varing the real parameter k, if the following linear system has solutions:

$$\begin{cases}x+y+z=k\\ kx+y+z=1\\ x+y+kz=0\end{cases}.$$

By Rouché-Capelli's Theorem the system has solution if and only if complete and incomplete matrices have the same rank, to calculate their ranks we reduce the matrices by elementary operations on their lines:

$$\begin{bmatrix}1 & 1 & 1 & | & k\\ k & 1 & 1 & | & 1\\ 1 & 1 & k & | & 0\end{bmatrix}\begin{matrix}\mathcal{R}_2\mapsto\mathcal{R}_2-k\mathcal{R}_1\\ \mathcal{R}_3\mapsto\mathcal{R}_3-\mathcal{R}_1\end{matrix}\begin{bmatrix}1 & 1 & 1 & | & k\\ 0 & 1-k & 1-k & | & 1-k^2\\ 0 & 0 & k-1 & | & -k\end{bmatrix}.$$

From the reduced matrices we note that if $k\neq 1$ both matrices (complete and incomplete) have rank equal 3, the system has only one solution; if $m=1$ the reduced matrices are $\begin{bmatrix}1 & 1 & 1 & | & 1\\ 0 & 0 & 0 & | & 0\\ 0 & 0 & 0 & | & -1\end{bmatrix}$, in this case the rank of the complete matrix is 2, while the incomplete has its rank equal 1; system hasn't solutions.

II M 1) Given the system $\begin{cases}f(x,y,z)=3xyz-x^2-y^2-z^2=0\\ g(x,y,z)=2e^{x-y}-e^{y-z}-e^{z-x}=0\end{cases}$ satisfied at the point $P(1,1,1)$, verify that it is possible to define an implicit function $x\to(y(x),z(x))$ and then calculate the derivatives of such function at $x=1$.

$\begin{cases}f(P)=3-1-1-1=0\\ g(P)=2-1-1=0\end{cases}$, conditions are satisfied in point P.

$\mathcal{J}(f,g) = \begin{bmatrix} 3yz - 2x & 3xz - 2y & 3xy - 2z \\ 2e^{x-y} + e^{z-x} & -2e^{x-y} - e^{y-z} & e^{y-z} - e^{z-x} \end{bmatrix}$,

$\mathcal{J}(f,g)(P) = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -3 & 0 \end{bmatrix}$; the determinant of the Jacobian restricted at the dependent variables is $\left|\mathcal{J}(f,g)(P)\big|_{y,z}\right| = \begin{vmatrix} 1 & 1 \\ -3 & 0 \end{vmatrix} = 3 \neq 0$; the proposed conditions define an implicit function $x \to (y(x), z(x))$.

$$y'(1) = -\frac{\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & 0 \end{vmatrix}} = -\frac{-3}{3} = 1\,;\; z'(1) = -\frac{\begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & 0 \end{vmatrix}} = -\frac{6}{3} = -2\,.$$

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y,z) = x + 3y + 6z \\ \text{u.c.: } x^2 + y^2 + z^2 = 46 \end{cases}$.

The function f is a polynomial, continuos function, the admissible region is a ball with center $(0,0,0)$ and radius $\sqrt{46}$, a bounded and closed set, therefore f presents absolute maximum and minimum in the admissible region,. The Lagrangian function is $\mathcal{L}(x,y,z,\lambda) = x + 3y + 6z - \lambda(x^2 + y^2 + z^2 - 46)$ with $\nabla\mathcal{L} = (1 - 2\lambda x, 3 - 2\lambda y, 6 - 2\lambda z, -(x^2 + y^2 + z^2 - 46))$.

$$FOC\colon \begin{cases} 1 - 2\lambda x = 0 \\ 3 - 2\lambda y = 0 \\ 6 - 2\lambda z = 0 \\ x^2 + y^2 + z^2 = 46 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{3}{2\lambda} \\ z = \frac{6}{2\lambda} \\ \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 + \left(\frac{6}{2\lambda}\right)^2 = 46 \end{cases} \Rightarrow$$

$$\begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{3}{2\lambda} \\ z = \frac{6}{2\lambda} \\ \frac{1}{4\lambda^2} + \frac{9}{4\lambda^2} + \frac{36}{4\lambda^2} = 46 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{3}{2\lambda} \\ z = \frac{6}{2\lambda} \\ \frac{46}{4\lambda^2} = 46 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{3}{2\lambda} \\ z = \frac{6}{2\lambda} \\ \lambda^2 = \frac{1}{4} \end{cases} \Rightarrow \begin{cases} x = \pm 1 \\ y = \pm 3 \\ z = \pm 6 \\ \lambda = \pm \frac{1}{2} \end{cases}$$

. Two constrained critical points $P_{1,2} = (\pm 1, \pm 3, \pm 6)$.

$SOC\colon \overline{\mathcal{H}} = \begin{bmatrix} 0 & 2x & 2y & 2z \\ 2x & -2\lambda & 0 & 0 \\ 2y & 0 & -2\lambda & 0 \\ 2z & 0 & 0 & -2\lambda \end{bmatrix}$; the proposed problem has three variables and one equality constraint, thus two border Hessian's principal minors are relevant, the third and the fourth: $\overline{\mathcal{H}}_3 = \begin{vmatrix} 0 & 2x & 2y \\ 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \end{vmatrix} =$

$$-2x\begin{vmatrix} 2x & 0 \\ 2y & -2\lambda \end{vmatrix} + 2y\begin{vmatrix} 2x & -2\lambda \\ 2y & 0 \end{vmatrix} = 8\lambda x^2 + 8\lambda y^2 = 8\lambda(x^2 + y^2);$$

$$\overline{\mathcal{H}}_4 = \begin{vmatrix} 0 & 2x & 2y & 2z \\ 2x & -2\lambda & 0 & 0 \\ 2y & 0 & -2\lambda & 0 \\ 2z & 0 & 0 & -2\lambda \end{vmatrix} =$$

$$-2x\begin{vmatrix} 2x & 0 & 0 \\ 2y & -2\lambda & 0 \\ 2z & 0 & -2\lambda \end{vmatrix} + 2y\begin{vmatrix} 2x & -2\lambda & 0 \\ 2y & 0 & 0 \\ 2z & 0 & -2\lambda \end{vmatrix} - 2z\begin{vmatrix} 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \\ 2z & 0 & 0 \end{vmatrix} =$$

$$4\lambda x\begin{vmatrix} 2x & 0 \\ 2y & -2\lambda \end{vmatrix} - 4\lambda y\begin{vmatrix} 2x & -2\lambda \\ 2y & 0 \end{vmatrix} - 4\lambda z\begin{vmatrix} 2y & -2\lambda \\ 2z & 0 \end{vmatrix} =$$

$-16\lambda^2x^2-16\lambda^2y^2-16\lambda^2z^2=-16\lambda^2(x^2+y^2+z^2)<0$ in points $P_{1,2}$. For the third principal minor $\overline{\mathcal{H}}_3(P_1)=40>0$, while $\overline{\mathcal{H}}_3(P_2)=-40<0$. P_1 is point of maximum, P_2 is point of minimum; $maxf=46$, $minf=-46$.

II M 3) Study the nature of the four critical points of the function
$f(x,y)=x^2+\dfrac{1}{x^2}+y^2+\dfrac{1}{y^2}$.

$\nabla f=\left(2x-\dfrac{2}{x^3},2y-\dfrac{2}{y^3}\right)$.

FOC: $\begin{cases}2x-\frac{2}{x^3}=0\\ 2y-\frac{2}{y^3}=0\end{cases}\Rightarrow\begin{cases}\frac{2x^4-2}{x^3}=0\\ \frac{2y^4-2}{y^3}=0\end{cases}\Rightarrow\begin{cases}2x^4-2=0\\ 2y^4-2=0\end{cases}\Rightarrow\begin{cases}x^4=1\\ y^4=1\end{cases}\Rightarrow$
$\begin{cases}x=\pm1\\ y=\pm1\end{cases}$. Four critical points: $P_1=(1,1)$, $P_2=(1,-1)$, $P_3=(-1,1)$, $P_4=(-1,-1)$.

SOC: $\mathcal{H}_f=\begin{bmatrix}2+\frac{6}{x^4} & 0\\ 0 & 2+\frac{6}{y^4}\end{bmatrix}$, $\mathcal{H}_f(P_{1,2,3,4})=\begin{bmatrix}8&0\\0&8\end{bmatrix}$, $\mathcal{H}_f^1=8>0$, and
$\mathcal{H}_f^2=\begin{vmatrix}8&0\\0&8\end{vmatrix}=64>0$. All the critical points are points of minimum, $minf=4$.

II M 4) Given the function $f(x,y)=x^2+y^2$, the two unit vectors $\boldsymbol{u}=\left(\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)$ and $\boldsymbol{v}=\left(\dfrac{\sqrt{2}}{2},-\dfrac{\sqrt{2}}{2}\right)$ and the point $P(1,1)$; calculate the directional derivatives $\mathcal{D}_{\boldsymbol{u}}f(P)$ and $\mathcal{D}_{\boldsymbol{v}}f(P)$ and find the unit vector $\boldsymbol{w}=\left(\dfrac{|w_1|}{\sqrt{w_1^2+w_2^2}},\dfrac{|w_2|}{\sqrt{w_1^2+w_2^2}}\right)$ such that $\mathcal{D}^{(2)}_{\boldsymbol{u},\boldsymbol{w}}f(P)+\mathcal{D}^{(2)}_{\boldsymbol{v},\boldsymbol{w}}f(P)=0$.

Function f is twice differentiable in all points of $\mathbb{R}^2$, by the differentiability of f, given unit vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ and point P, $\mathcal{D}_{\boldsymbol{x}}f(P)=\nabla f(P)\cdot\boldsymbol{x}$ and
$\mathcal{D}^{(2)}_{\boldsymbol{x},\boldsymbol{y}}f(P)=\boldsymbol{x}^T\cdot\mathcal{H}_f(P)\cdot\boldsymbol{y}$.

$\nabla f(x,y)=(2x,2y)$, $\nabla f(P)=(2,2)$;

$\mathcal{D}_{\boldsymbol{u}}f(P)=(2,2)\cdot\left(\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)=\sqrt{2}+\sqrt{2}=2\sqrt{2}$,

$\mathcal{D}_{\boldsymbol{v}}f(P)=(2,2)\cdot\left(\dfrac{\sqrt{2}}{2},-\dfrac{\sqrt{2}}{2}\right)=\sqrt{2}-\sqrt{2}=0$.

$\mathcal{H}_f=\begin{bmatrix}2&0\\0&2\end{bmatrix}$; $\mathcal{D}^{(2)}_{\boldsymbol{u},\boldsymbol{w}}f(P)+\mathcal{D}^{(2)}_{\boldsymbol{v},\boldsymbol{w}}f(P)=$

$\left(\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)\cdot\begin{bmatrix}2&0\\0&2\end{bmatrix}\cdot\begin{pmatrix}\frac{|w_1|}{\sqrt{w_1^2+w_2^2}}\\ \frac{|w_2|}{\sqrt{w_1^2+w_2^2}}\end{pmatrix}+\left(\dfrac{\sqrt{2}}{2},-\dfrac{\sqrt{2}}{2}\right)\cdot\begin{bmatrix}2&0\\0&2\end{bmatrix}\cdot\begin{pmatrix}\frac{|w_1|}{\sqrt{w_1^2+w_2^2}}\\ \frac{|w_2|}{\sqrt{w_1^2+w_2^2}}\end{pmatrix}=$

$\left(\sqrt{2},\sqrt{2}\right)\cdot\left(\dfrac{|w_1|}{\sqrt{w_1^2+w_2^2}},\dfrac{|w_2|}{\sqrt{w_1^2+w_2^2}}\right)+$

$\left(\sqrt{2},-\sqrt{2}\right)\cdot\left(\dfrac{|w_1|}{\sqrt{w_1^2+w_2^2}},\dfrac{|w_2|}{\sqrt{w_1^2+w_2^2}}\right)=$

$\frac{(|w_1|+|w_2|)\sqrt{2}}{\sqrt{w_1^2+w_2^2}}+\frac{(|w_1|-|w_2|)\sqrt{2}}{\sqrt{w_1^2+w_2^2}} = \frac{2|w_1|\sqrt{2}}{\sqrt{w_1^2+w_2^2}}$. Put $\frac{2|w_1|\sqrt{2}}{\sqrt{w_1^2+w_2^2}} = 0$, we get $w_1 = 0$ and $\boldsymbol{w} = \left(\frac{0}{\sqrt{0^2+w_2^2}}, \frac{|w_2|}{\sqrt{0^2+w_2^2}}\right) = \left(0, \frac{|w_2|}{\sqrt{w_2^2}}\right) = \left(0, \frac{|w_2|}{|w_2|}\right) = (0, 1)$.