UNIVERSITÁ DEGLI STUDI DI SIENA
Scuola di Economia e Management
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Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 11/6/2026

I M 1) Find the roots of the equation $x^4+1=0$ and then calculate the sum and the product of the two roots with positive real part.

From the equation $x^4+1=0$, we get $x^4=-1$ and $x=\sqrt[4]{-1}$; remember that $-1=cos\,\pi+i\,sin\,\pi$ thus $x=\sqrt[4]{cos\,\pi+i\,sin\,\pi}$ and by De Moivre's formula $x=cos\left(\frac{\pi+2k\pi}{4}\right)+i\,sin\left(\frac{\pi+2k\pi}{4}\right)\ k=0,1,2,3.$

The four are:

$k=0\rightarrow\ x_0=cos\frac{\pi}{4}+i\,sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$;

$k=1\rightarrow\ x_1=cos\frac{3\pi}{4}+i\,sin\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$;

$k=2\rightarrow\ x_2=cos\frac{5\pi}{4}+i\,sin\frac{5\pi}{4}=-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$;

$k=3\rightarrow\ x_3=cos\frac{7\pi}{4}+i\,sin\frac{7\pi}{4}=\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$.

The two roots with positive real parts are x_0 and x_3, their sum is $x_0+x_3=\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=\sqrt{2}$ while their product is $x_0\cdot x_3=\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=\frac{1}{2}-\frac{1}{2}i^2=1$.

I M 2) Consider the matrix $\mathbb{A}=\begin{bmatrix}1&1&1&1\\0&1&0&0\\0&0&1&0\\1&1&1&1\end{bmatrix}$. Calculate its eigenvalues and study if the matrix $\mathbb{A}$ is a diagonalizable one.

The first step is the calculus of the characteristic polynomial of the matrix:

$P_{\mathbb{A}}(\lambda)=|\lambda\mathbb{I}-\mathbb{A}|=\begin{vmatrix}\lambda-1&-1&-1&-1\\0&\lambda-1&0&0\\0&0&\lambda-1&0\\-1&-1&-1&\lambda-1\end{vmatrix}=$

$\begin{vmatrix}\lambda-1&0\\0&\lambda-1\end{vmatrix}\cdot\begin{vmatrix}\lambda-1&-1\\-1&\lambda-1\end{vmatrix}=(\lambda-1)^2\left((\lambda-1)^2-1\right)=\lambda(\lambda-1)^2(\lambda-2)$. Put $P_{\mathbb{A}}(\lambda)=0$ we find an eigenvalue $\lambda=1$ with algebraic multiplicity equal two and the eigenvalues $\lambda=0$ and $\lambda=2$ with algebraic multiplicity equal one. The matrix $\mathbb{A}$ is diagonalizable if and only if the geometric multiplicity of the eigenvalue $\lambda=1$ is two. To find the geometric multiplicity of the eigenvalue $\lambda=1$ we calculate the rank of the matrix $1 \cdot \mathbb{I} - \mathbb{A} = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix}$; note that two raws of the matrix are null and the 2 by 2 submatrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ has determinant different from zero. We conclude that rank of matrix $1 \cdot \mathbb{I} - \mathbb{A}$ is two and the geometric multiplicity of the eigenvalue $\lambda = 1$ is $4 - Rank(1 \cdot \mathbb{I} - \mathbb{A}) = 2$; matrix $\mathbb{A}$ is diagonalizable.

I M 3) Given a linear map $F: \mathbb{R}^3 \to \mathbb{R}^4$, with $F(x_1, x_2, x_3) = (0, ax_1, bx_1 + x_2, cx_1 + x_2 + x_3)$; we know that $F(1,1,1) = (0,2,4,8)$. Calculate the values of the parameters a, b and c and then find a basis for the image of linear map F.

$F(1,1,1) = (0, a, b+1, c+2)$, put $(0, a, b+1, c+2) = (0,2,4,8)$ easily follows $a = 2$, $b = 3$ and $c = 6$, the linear map F is $F(x_1, x_2, x_3) = (0, 2x_1, 3x_1 + x_2, 6x_1 + x_2 + x_3)$. To find a basis for the image of linear map F we have that a generic element of the image is $(0, 2x_1, 3x_1 + x_2, 6x_1 + x_2 + x_3) = x_1(0,2,3,6) + x_2(0,0,1,1) + x_3(0,0,0,1)$, a basis for the image is the set $\mathcal{B}_{IM(F)} = \{(0,2,3,6), (0,0,1,1), (0,0,0,1)\}$.

I M 4) Given the matrix $\mathbb{X} = \begin{bmatrix} \frac{1}{2} & \alpha \\ \alpha & -\frac{1}{2} \end{bmatrix}$, find the values of parameter α knowing that matrix $\mathbb{X}$ is an orthogonal matrix, and with the values of α found calculate the determinant of matrix $\mathbb{X}$.

Matrix $\mathbb{X}$ is an orthogonal matrix if $\mathbb{X}^T \cdot \mathbb{X} = \mathbb{X} \cdot \mathbb{X} = \mathbb{I}$.

$\mathbb{X} \cdot \mathbb{X} = \begin{bmatrix} \frac{1}{2} & \alpha \\ \alpha & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \alpha \\ \alpha & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} + \alpha^2 & 0 \\ 0 & \frac{1}{4} + \alpha^2 \end{bmatrix}$ and matrix $\mathbb{X}$ is an orthogonal matrix if and only if $\frac{1}{4} + \alpha^2 = 1$, with solution $\alpha = \pm \frac{1}{2}\sqrt{3}$. The determinant of the matrix is $\begin{vmatrix} \frac{1}{2} & \pm \frac{1}{2}\sqrt{3} \\ \pm \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \left(\pm \frac{1}{2}\sqrt{3} \right)^2 = -\frac{1}{4} - \frac{3}{4} = -1$.

II M 1) Given the equation $f(x,y,z) = e^{x+y-z} - 2xyz - z = 0$ satisfied at the point $(1,0,1)$, verify that with it an implicit function $y = y(x,z)$ can be defined and then calculate, for this implicit function its gradient vector $\nabla y(1,1)$.

$f(1,0,1) = e^0 - 0 - 1 = 0$, $f'_x = e^{x+y-z} - 2yz$, $f'_y = e^{x+y-z} - 2xz$ and $f'_z = -e^{x+y-z} - 2xy - 1$, with $f'_x(1,0,1) = 1$, $f'_y(1,0,1) = -1$ and $f'_z(1,0,1) = -2$. Since $f'_y(0,0,0) \neq 0$, the equation $f(x,y,z) = e^{x+y-z} - 2xyz - z = 0$ defines a function $y = y(x,z)$ with $\nabla y = (y'_x(1,1), y'_z(1,1)) = \left(-\frac{f'_x(1,0,1)}{f'_y(1,0,1)}, -\frac{f'_z(1,0,1)}{f'_y(1,0,1)} \right) = (1, -2)$.

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x + xy \\ \text{u.c.: } x^2 + 4y^2 \leq 4 \end{cases}$.

The function f is a polynomial, continuos function, the admissible region is an ellipse with center $(0,0)$, a bounded and closed set, see figure in the next page; therefore f presents absolute maximum and minimum in the admissible region. The Lagrangian function is

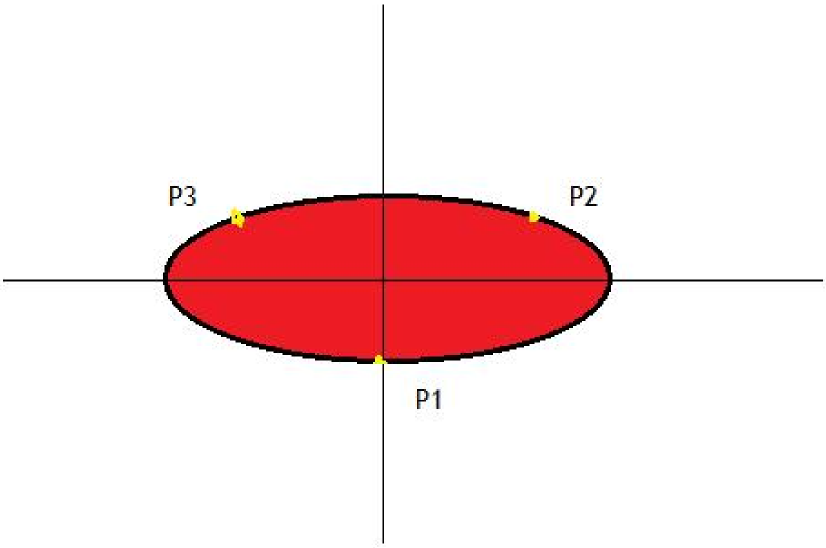
$\mathcal{L}(x,y,\lambda) = x + xy - \lambda(x^2 + 4y^2 - 4)$ with
$\nabla\mathcal{L} = (1 + y - 2\lambda x, x - 8\lambda y, -(x^2 + 4y^2 - 4))$.
$I^\circ\ CASE\ (free\ optimization)$:

$\begin{cases} \lambda = 0 \\ 1 + y = 0 \\ x = 0 \\ x^2 + 4y^2 \le 4 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ y = -1 \\ x = 0 \\ 0^2 + 4(-1)^2 \le 4 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ y = -1 \\ x = 0 \\ 4 \le 4 \end{cases}$; point P_1 is a boundary admissible point, $\mathcal{H}_f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the second principal minor of the hessian matrix is $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$; P_1 is a free saddle point.

$II^\circ\ CASE\ (constrained\ optimization)$:

$\begin{cases} \lambda \ne 0 \\ 1 + y - 2\lambda x = 0 \\ x - 8\lambda y = 0 \\ x^2 + 4y^2 = 4 \end{cases}$, we can observe that if $y = 0$ then $x = 0$ and the last constraint isn't true, so $y \ne 0$ and $\lambda = \dfrac{x}{8y}$; put the obtained λ in the system we have:

$\begin{cases} \lambda \ne 0 \\ 4y + 4y^2 - x^2 = 0 \\ \lambda = \frac{x}{8y} \\ x^2 = 4 - 4y^2 \end{cases} \Rightarrow \begin{cases} \lambda \ne 0 \\ 8y^2 + 4y - 4 = 0 \\ \lambda = \frac{x}{8y} \\ x^2 = 4 - 4y^2 \end{cases}$. The equation $8y^2 + 4y - 4 = 0$ has solutions $y = -1 \vee y = \dfrac{1}{2}$; if $y = -1$ then $x = 0$ and $\lambda = 0$, solution not accepted; if $y = \dfrac{1}{2}$ then $x^2 = 3$, $x = \pm\sqrt{3}$ and $\lambda = \pm\dfrac{1}{4}\sqrt{3}$. Point $P_2 = \left(\sqrt{3}, \dfrac{1}{2}\right)$ is point of maximum ($\lambda > 0$) while point $P_3 = \left(-\sqrt{3}, \dfrac{1}{2}\right)$ is point of minimum ($\lambda < 0$), f presents absolute maximum equal $\dfrac{3}{2}\sqrt{3}$ in point P_2 and absolute minimum equal $-\dfrac{3}{2}\sqrt{3}$ in point P_3.



II M 3) Given the vector value function $f\colon \mathbb{R} \to \mathbb{R}^3$, $t \to (2t, 1+t^3, sin\, t + cos\, t)$, determine the equation of the tangent line to this curve at the point $t = 0$.
The equation of tangent line in parametric form is $r(t) = f(0) + f'(0) \cdot t$;
$f(0) = (0, 1, 1)$, $f'(t) = (2, 3t^2, cos\, t - sin\, t)$, $f'(0) = (2, 0, 1)$ and
$r(t) = (0, 1, 1) + (2, 0, 1) \cdot t = (2t, 1, 1+t)$.

II M 4) Given the function $f(x, y) = (x+y)e^{x+y}$ and the unit vector $v = (cos\, \alpha, sin\, \alpha)$, determine the values of α for which the directional derivative $\mathcal{D}_v f(0,0)$ is equal to zero and then, for any found α, calculate $\mathcal{D}^{(2)}_{v,v} f(0,0)$.
Function f is twice differentiable in all points of $\mathbb{R}^2$ and is an exchangeable variables function ($f(x, y) = f(y, x),\ \forall (x, y) \in \mathbb{R}^2$), thus $f'_x = f'_y$ and $f''_{xx} = f''_{yy}$;
$f'_x = e^{x+y} + (x+y)e^{x+y} = (1+x+y)e^{x+y}$ with $f'_x(0,0) = 1$ and
$\mathcal{D}_v f(0,0) = \nabla f(0,0) \cdot v = (1, 1) \cdot (cos\, \alpha, sin\, \alpha) = cos\, \alpha + sin\, \alpha$, put
$cos\, \alpha + sin\, \alpha = 0$ or $tan\, \alpha = -1$ we get $\alpha = \dfrac{3\pi}{4} \vee \alpha = \dfrac{7\pi}{4}$.
$f''_{xx} = e^{x+y} + (1+x+y)e^{x+y} = (2+x+y)e^{x+y}$ with $f''_{xx}(0,0) = 2$,
$f''_{xy} = e^{x+y} + (1+x+y)e^{x+y} = (2+x+y)e^{x+y}$ with $f''_{xy}(0,0) = 2$, and the second order directional derivative $\mathcal{D}^{(2)}_{v,v} f(0,0)$ is calculated by the product $v^T \cdot \mathcal{H}_f(0,0) \cdot v =$
$(cos\, \alpha, sin\, \alpha) \cdot \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \cdot \begin{pmatrix} cos\, \alpha \\ sin\, \alpha \end{pmatrix} = (cos\, \alpha, sin\, \alpha) \cdot \begin{pmatrix} 2(cos\, \alpha + sin\, \alpha) \\ 2(cos\, \alpha + sin\, \alpha) \end{pmatrix} =$
$2(cos^2\, \alpha + cos\, \alpha\, sin\, \alpha) + 2(sin\, \alpha\, cos\, \alpha + sin^2\, \alpha) =$
$2(cos^2\, \alpha + 2\, cos\, \alpha\, sin\, \alpha + sin^2\, \alpha) = 2(cos\, \alpha + sin\, \alpha)^2 = 0$.