UNIVERSITÁ DEGLI STUDI DI SIENA
Scuola di Economia e Management
A.A. 2025/26
Quantitative Methods for Economic Applications - Mathematics for Economic Applications
Task 29/6/2026

I M 1) Calculate $\sqrt{(1+i)^4+(1-i)^4}$.

Complex number $1+i$ has modulus equal $\sqrt{2}$ and argument $\frac{\pi}{4}$ while complex number $1-i$ has modulus equal $\sqrt{2}$ and argument $-\frac{\pi}{4}$, thus the two complex number can be written in exponential form as $\sqrt{2}e^{\frac{\pi}{4}i}$ and $\sqrt{2}e^{-\frac{\pi}{4}i}$; substitute in the square root we get

$\sqrt{(1+i)^4+(1-i)^4} = \sqrt{\left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^4+\left(\sqrt{2}e^{-\frac{\pi}{4}i}\right)^4} = \sqrt{4e^{\pi i}+4e^{-\pi i}} =$

$\sqrt{4(cos\,\pi + i\,sin\,\pi + cos(-\pi) + i\,sin(-\pi))} =$

$\sqrt{4(cos\,\pi + i\,sin\,\pi + cos\,\pi - i\,sin\,\pi)} = \sqrt{8\,cos\,\pi} = \sqrt{-8}$; remember that $-8 = 8(cos\,\pi + i\,sin\,\pi)$ thus by De Moivre's formula the two roots are

$z_k = \sqrt{8}\,cos\left(\frac{\pi+2k\pi}{2}\right) + i\,sin\left(\frac{\pi+2k\pi}{2}\right) =$

$2\sqrt{2}\,cos\left(\frac{\pi}{2}+k\pi\right) + i\,sin\left(\frac{\pi}{2}+k\pi\right)\ \ k = 0, 1.$

The roots are:

$k = 0 \rightarrow\ z_0 = 2\sqrt{2}\left(cos\frac{\pi}{2} + i\,sin\frac{\pi}{2}\right) = 2\sqrt{2}\,i\,;$

$k = 1 \rightarrow\ z_1 = 2\sqrt{2}\left(cos\frac{3}{2}\pi + i\,sin\frac{3}{2}\pi\right) = -2\sqrt{2}\,i\,.$

ALTERNATIVE SOLUTION: apply Pascal's Triangle Rule $(1+i)^4+(1-i)^4 = (1+4i+6i^2+4i^3+i^4)+(1-4i+6i^2-4i^3+i^4) = 2+12i^2+2i^4 = 2-12+2 = -8$; the rest of the exercise follows as above.

I M 2) Given the matrix $\mathbb{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Determine its eigenvalues and the corresponding eigenvectors.

The first step is the calculus of the characteristic polynomial of the matrix:

$P_{\mathbb{A}}(\lambda) = |\lambda\mathbb{I} - \mathbb{A}| = \begin{vmatrix} \lambda-1 & -1 & -1 \\ 0 & \lambda-2 & 0 \\ -1 & -1 & \lambda-1 \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix} =$

$(\lambda-2)\left((\lambda-1)^2-1\right) = \lambda(\lambda-2)^2$. Put $P_{\mathbb{A}}(\lambda) = 0$ we find an eigenvalue $\lambda = 2$ with algebraic multiplicity equal two and the eigenvalue $\lambda = 0$ with algebraic multiplicity equal one. The eigenvectors corresponding to the eigenvalue $\lambda = 2$ are the vectors $\boldsymbol{v} = (v_1, v_2, v_3)$ that satisfy $\mathbb{A}\cdot\boldsymbol{v} = 2\cdot\boldsymbol{v} \Leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 2\cdot\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \Leftrightarrow$

$\begin{pmatrix} v_1+v_2+v_3 \\ 2v_2 \\ v_1+v_2+v_3 \end{pmatrix} = \begin{pmatrix} 2v_1 \\ 2v_2 \\ 2v_3 \end{pmatrix} \Rightarrow (v_2 = 0 \wedge v_3 = v_1)$. A eigenvector corresponding to the eigenvalue $\lambda = 2$ is $\boldsymbol{v} = (v_1, 0, v_1) = v_1(1, 0, 1)$. Note that the eigenspace associated to the eigenvalue $\lambda = 2$ has dimention 1. The eigenvectors corresponding to the eigenvalue $\lambda = 0$ are the vectors $\boldsymbol{v} = (v_1, v_2, v_3)$ that satisfy

$$\mathbb{A} \cdot \boldsymbol{v} = 0 \cdot \boldsymbol{v} \Leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} v_1 + v_2 + v_3 \\ 2v_2 \\ v_1 + v_2 + v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow (v_2 = 0 \wedge v_3 = -v_1).$$

A eigenvector corresponding to the eigenvalue $\lambda = 0$ is $\boldsymbol{v} = (v_1, 0, -v_1) = v_1(1, 0, -1)$.

I M 3) Given a linear map $F: \mathbb{R}^3 \to \mathbb{R}^4$, with $F(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_1 - x_3, x_1 + 3x_2 + 2x_3)$. Calculate the dimention of the Image and the dimention of the Kernel of F; and for both, Image and Kernel, find a basis.

The matrix associated to the linear map is $\mathbb{A}_F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 3 & 2 \end{bmatrix}$; now we reduce the matrix by linear operation on its lines:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 3 & 2 \end{bmatrix} \begin{matrix} R_3 \mapsto R_3 - R_1 \\ R4 \mapsto R_4 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{matrix} R_3 \mapsto R_3 + R_2 \\ R4 \mapsto R_4 - 2R_2 \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

. Matrix $\mathbb{A}_F$ has rank 2, thus the dimention of the Image of F is 2 and by the Rank-Nullity Theorem the dimention of the Kernel is $dim(KerF) = dim(\mathbb{R}^3) - dim(ImF) = 3 - 2 = 1$.

For a basis of the image we can consider a generic element of the image as the vector (y_1, y_2, y_3, y_4); put $(y_1, y_2, y_3, y_4) = F(x_1, x_2, x_3)$ we achive the linear system:

$$\begin{cases} x_1 + x_2 = y_1 \\ x_2 + x_3 = y_2 \\ x_1 - x_3 = y_3 \\ x_1 + 3x_2 + 2x_3 = y_4 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 = y_1 \\ x_2 + x_1 - y_3 = y_2 \\ x_3 = x_1 - y_3 \\ x_1 + 3x_2 + 2(x_1 - y_3) = y_4 \end{cases} \Rightarrow$$

$$\begin{cases} x_1 + x_2 = y_1 \\ x_1 + x_2 = y_2 + y_3 \\ x_3 = x_1 - y_3 \\ 3(x_1 + x_2) = 2y_3 + y_4 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 = y_1 \\ y_1 = y_2 + y_3 \\ x_3 = x_1 - y_3 \\ 3y_1 = 2y_3 + y_4 \end{cases} \Rightarrow \begin{cases} y_2 = y_3 - y_1 \\ y_4 = 2y_3 - 3y_1 \end{cases}$$

. A generic element of the image is the vector $(y_1, y_2, y_3, y_4) = (y_1, y_3 - y_1, y_3, 2y_3 - 3y_1) = y_1(1, -1, 0, -3) + y_3(0, 1, 1, 2)$, a basis for the image is the set $\mathcal{B}_{Im(F)} = \{(1, -1, 0, -3), (0, 1, 1, 2)\}$.

For a basis of the kernel we know that a vector (x_1, x_2, x_3) belongs in the kernel of F if $F(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_1 - x_3, x_1 + 3x_2 + 2x_3) = (0, 0, 0, 0)$; write the condition in linear system form we get:

$$\begin{cases} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \\ x_1 - x_3 = 0 \\ x_1 + 3x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = -x_1 \\ x_3 = x_1 \end{cases}$$

. A generic element of the kernel is the vector $(x_1, x_2, x_3) = (x_1, -x_1, x_1) = x_1(1, -1, 1)$, a basis for the kernel is the set $\mathcal{B}_{Ker(F)} = \{(1, -1, 1)\}$.

ALTERNATIVE SOLUTION: the dimention and a basis for the image of linear application can be calculated jointly, if $\boldsymbol{z} = (z_1, z_2, z_3, z_4)$ is a vector that belongs in the codomain of F, $\mathbb{R}^4$, $\boldsymbol{z} \in ImF$ if and only if $F(x_1,x_2,x_3) = (x_1+x_2, x_2+x_3, x_1-x_3, x_1+3x_2+2x_3) = (z_1,z_2,z_3,z_4)$, now we write the condition in matrix form: $\begin{bmatrix} x_1 & x_2 & 0 & | & z_1 \\ 0 & x_2 & x_3 & | & z_2 \\ x_1 & 0 & -x_3 & | & z_3 \\ x_1 & 3x_2 & 2x_3 & | & z_4 \end{bmatrix}$ and we reduce it by linear operation on its lines: $\begin{bmatrix} x_1 & x_2 & 0 & | & z_1 \\ 0 & x_2 & x_3 & | & z_2 \\ x_1 & 0 & -x_3 & | & z_3 \\ x_1 & 3x_2 & 2x_3 & | & z_4 \end{bmatrix} \begin{matrix} R_3 \mapsto R_3 - R_1 \\ R4 \mapsto R_4 - R_1 \end{matrix}$

$$\begin{bmatrix} x_1 & x_2 & 0 & | & z_1 \\ 0 & x_2 & x_3 & | & z_2 \\ 0 & -x_2 & -x_3 & | & z_3 - z_1 \\ 0 & 2x_2 & 2x_3 & | & z_4 - z_1 \end{bmatrix} \begin{matrix} R_3 \mapsto R_3 + R_2 \\ R4 \mapsto R_4 - 2R_2 \end{matrix} \begin{bmatrix} x_1 & x_2 & 0 & | & z_1 \\ 0 & x_2 & x_3 & | & z_2 \\ 0 & 0 & 0 & | & z_3 - z_1 + z_2 \\ 0 & 0 & 0 & | & z_4 - z_1 - 2z_2 \end{bmatrix}.$$

From the last matrix we note that complete and incomplete matrices have equal rank if and only if $z_3 = z_1 - z_2$ and $z_4 = z_1 + 2z_2$, the common rank is 2 and a generic vector that belongs in the image is $(z_1,z_2,z_3,z_4) = (z_1,z_2,z_1-z_2,z_1+2z_2) = z_1(1,0,1,1) + z_2(0,1,-1,2)$; the dimention of the image is 2 and a basis for the image is the set $\mathcal{B}_{Im(F)} = \{(1,0,1,1),(0,1,-1,2)\}$. The rest of the exercise follows as above.

I M 4) Verify that the matrix $\mathbb{X} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$ is similar to the matrix $\mathbb{Y} = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and find a matrix $\mathbb{Z}$ that realizes the similarity between $\mathbb{X}$ and $\mathbb{Y}$.

Matrix $\mathbb{X}$ is similar to the matrix $\mathbb{Y}$ if exist an invertible matrix $\mathbb{Z}$ such that $\mathbb{Y} = \mathbb{Z}^{-1} \cdot \mathbb{X} \cdot \mathbb{Z}$ or $\mathbb{Z} \cdot \mathbb{Y} = \mathbb{X} \cdot \mathbb{Z}$. Put $\mathbb{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$, premultiply $\mathbb{Y}$ by $\mathbb{Z}$ and postmultiply $\mathbb{X}$ by $\mathbb{Z}$ we obtain $\mathbb{Z} \cdot \mathbb{Y} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \cdot \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} =$ $\begin{bmatrix} \frac{5}{2}z_{11} - \frac{1}{2}z_{12} & -\frac{1}{2}z_{11} + \frac{1}{2}z_{12} \\ \frac{5}{2}z_{21} - \frac{1}{2}z_{22} & -\frac{1}{2}z_{21} + \frac{1}{2}z_{22} \end{bmatrix}$ and $\mathbb{X} \cdot \mathbb{Z} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} =$ $\begin{bmatrix} z_{21} & z_{22} \\ -z_{11} + 3z_{21} & -z_{12} + 3z_{22} \end{bmatrix}$; put $\begin{bmatrix} \frac{5}{2}z_{11} - \frac{1}{2}z_{12} & -\frac{1}{2}z_{11} + \frac{1}{2}z_{12} \\ \frac{5}{2}z_{21} - \frac{1}{2}z_{22} & -\frac{1}{2}z_{21} + \frac{1}{2}z_{22} \end{bmatrix} = \begin{bmatrix} z_{21} & z_{22} \\ -z_{11} + 3z_{21} & -z_{12} + 3z_{22} \end{bmatrix}$ and write the four conditions as a linear system $\begin{cases} \frac{5}{2}z_{11} - \frac{1}{2}z_{12} = z_{21} \\ -\frac{1}{2}z_{11} + \frac{1}{2}z_{12} = z_{22} \\ \frac{5}{2}z_{21} - \frac{1}{2}z_{22} = -z_{11} + 3z_{21} \\ -\frac{1}{2}z_{21} + \frac{1}{2}z_{22} = -z_{12} + 3z_{22} \end{cases} \Rightarrow$

$$\begin{cases} 5z_{11} - z_{12} = 2z_{21} \\ -z_{11} + z_{12} = 2z_{22} \\ 5z_{21} - z_{22} = -2z_{11} + 6z_{21} \\ -z_{21} + z_{22} = -2z_{12} + 6z_{22} \end{cases} \Rightarrow$$

$$\begin{cases} z_{12} = 5z_{11} - 2z_{21} \\ -z_{11} + 5z_{11} - 2z_{21} = 2z_{22} \\ z_{22} = 2z_{11} - z_{21} \\ -z_{21} + 2z_{11} - z_{21} = -2(5z_{11} - 2z_{21}) + 6(2z_{11} - z_{21}) \end{cases} \Rightarrow$$

$$\begin{cases} z_{12} = 5z_{11} - 2z_{21} \\ -z_{11} + 5z_{11} - 2z_{21} = 2(2z_{11} - z_{21}) \\ z_{22} = 2z_{11} - z_{21} \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} z_{12} = 5z_{11} - 2z_{21} \\ 0 = 0 \\ z_{22} = 2z_{11} - z_{21} \\ 0 = 0 \end{cases};$$

$\mathbb{Z} = \begin{bmatrix} z_{11} & 5z_{11} - 2z_{21} \\ z_{21} & 2z_{11} - z_{21} \end{bmatrix}$ with determinant $\begin{vmatrix} z_{11} & 5z_{11} - 2z_{21} \\ z_{21} & 2z_{11} - z_{21} \end{vmatrix} =$
$z_{11}(2z_{11} - z_{21}) - z_{21}(5z_{11} - 2z_{21}) = 2z_{11}^2 - 6z_{11}z_{21} + 2z_{21}^2$. Take, for instance, $z_{11} = z_{21} = 1$, the determinant is different from zero, matrix $\mathbb{Z} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ is invertible and we can conclude that $\mathbb{X}$ and $\mathbb{Y}$ are similar matrices.

II M 1) Given the equation $f(x,y,z) = x^2y - y^3z^2 + xyz = 1$ satisfied at the point $(-1,1,0)$, verify that with it an implicit function $z = z(x,y)$ can be defined and then calculate, for this implicit function, its gradient vector $\nabla z(-1,1)$.

$f(-1,1,0) = 1 - 0 - 0 = 1$, $f'_x = 2xy + yz$, $f'_y = x^2 - 3y^2z^2 + xz$ and $f'_z = -2y^3z + xy$, with $f'_x(-1,1,0) = -2$, $f'_y(-1,1,0) = 1$ and $f'_z(-1,1,0) = -1$. Since $f'_z(-1,1,0) \neq 0$, the equation $f(x,y,z) = e^{x+y-z} - 2xyz - z = 0$ defines a function $z = z(x,y)$ with $\nabla z = \left(z'_x(-1,1), z'_y(-1,1)\right) = \left(-\dfrac{f'_x(-1,1,0)}{f'_z(-1,1,0)}, -\dfrac{f'_y(-1,1,0)}{f'_z(-1,1,0)}\right) = (-2,1)$.

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = xy - y \\ \text{u.c.: } x^2 + 2y^2 \leq 1 \end{cases}$.

The function f is a polynomial, continuos function, the admissible region is an ellipse with center $(0,0)$, a bounded and closed set, see figure in the next page; therefore f presents absolute maximum and minimum in the admissible region. The Lagrangian function of the problem is
$\mathcal{L}(x,y,\lambda) = xy - y - \lambda(x^2 + 2y^2 - 1)$ with
$\nabla\mathcal{L} = (y - 2\lambda x, x - 1 - 4\lambda y, -(x^2 + 2y^2 - 1))$.

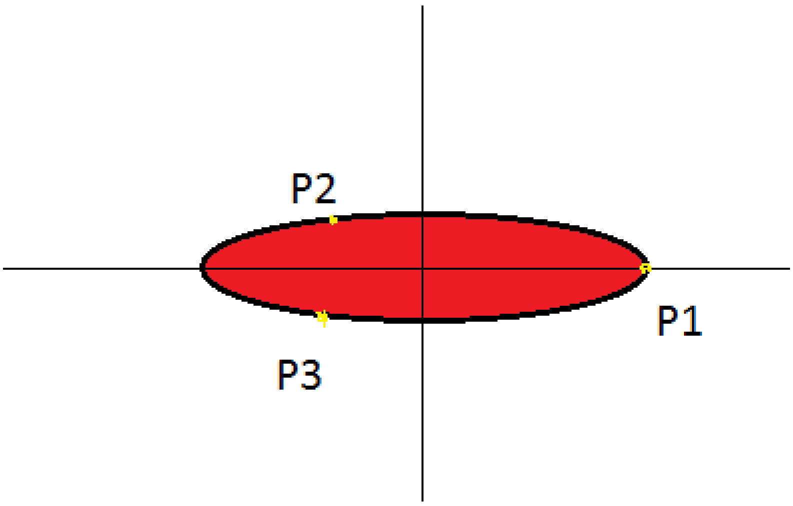
I° CASE (*free optimization*):

$\begin{cases} \lambda = 0 \\ y = 0 \\ x - 1 = 0 \\ x^2 + 2y^2 \leq 1 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ y = 0 \\ x = 1 \\ 1^2 + 2(0)^2 \leq 1 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ y = 0 \\ x = 1 \\ 1 \leq 1 \end{cases}$; point $P_1 = (1,0)$ is a boundary admissible point, $\mathcal{H}_f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the second principal minor of the hessian matrix is $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$; P_1 is a free saddle point.

II° CASE (*constrained optimization*):

$\begin{cases} \lambda \neq 0 \\ y - 2\lambda x = 0 \\ x - 1 - 4\lambda y = 0 \\ x^2 + 2y^2 = 1 \end{cases}$, we can observe that if $x = 0$ then $y = 0$ and the last constraint isn't true, so $x \neq 0$ and $\lambda = \dfrac{y}{2x}$; put the obtained λ in the system we have:

$\begin{cases} \lambda \neq 0 \\ \lambda = \frac{y}{2x} \\ 2x^2 - 2x - 4y^2 = 0 \\ 2y^2 = 1 - x^2 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ \lambda = \frac{y}{2x} \\ 4x^2 - 2x - 2 = 0 \\ y^2 = \frac{1-x^2}{2} \end{cases}$. The equation $4x^2 - 2x - 2 = 0$ has solutions $x = -\dfrac{1}{2} \vee x = 1$; if $x = 1$ then $y = 0$ and $\lambda = 0$, solution not accepted; if $x = -\dfrac{1}{2}$ then $y = \pm\dfrac{1}{4}\sqrt{6}$ and $\lambda = \mp\dfrac{1}{4}\sqrt{6}$. Point $P_2 = \left(-\dfrac{1}{2}, \dfrac{1}{4}\sqrt{6}\right)$ is point of minimum ($\lambda < 0$) while point $P_3 = \left(-\dfrac{1}{2}, -\dfrac{1}{4}\sqrt{6}\right)$ is point of maximum ($\lambda > 0$), f presents absolute minimum equal $-\dfrac{3}{8}\sqrt{6}$ in point P_2 and absolute maximum equal $\dfrac{3}{8}\sqrt{6}$ in point P_3.



II M 3) Find the point of minimum and the point of maximum for the function $f(x,y) = (x+y)e^{-(x^2+y^2)}$.
$\nabla f = \left(f'_x, f'_y\right)$, with $f'_x = 1 \cdot e^{-(x^2+y^2)} + (x+y)e^{-(x^2+y^2)}(-2x) = (1 - 2x(x+y))e^{-(x^2+y^2)}$ and $f'_y = 1 \cdot e^{-(x^2+y^2)} + (x+y)e^{-(x^2+y^2)}(-2y) = (1 - 2y(x+y))e^{-(x^2+y^2)}$.

FOC:

$\begin{cases} (1-2x(x+y))e^{-(x^2+y^2)} = 0 \\ (1-2y(x+y))e^{-(x^2+y^2)} = 0 \end{cases} \Rightarrow \begin{cases} 1-2x(x+y) = 0 \\ 1-2y(x+y) = 0 \end{cases}$; obviously $x+y \neq 0$, thus

$\begin{cases} 2x = \frac{1}{x+y} \\ 2y = \frac{1}{x+y} \end{cases} \Rightarrow \begin{cases} 2x = \frac{1}{x+y} \\ y = x \end{cases} \Rightarrow \begin{cases} 4x^2 = 1 \\ y = x \end{cases} \Rightarrow \begin{cases} x = \pm\frac{1}{2} \\ y = \pm\frac{1}{2} \end{cases}$. Two critical points

$P_{12} = \left(\pm\dfrac{1}{2}, \pm\dfrac{1}{2}\right)$.

SOC:

$f''_{xx} = (-4x-2y)e^{-(x^2+y^2)} + (1-2x(x+y))e^{-(x^2+y^2)}(-2x) =$
$(4x^3+4x^2y-6x-2y)e^{-(x^2+y^2)}$;

$f''_{yy} = (-2x-4y)e^{-(x^2+y^2)} + (1-2y(x+y))e^{-(x^2+y^2)}(-2y) =$
$(4xy^2+4y^3-2x-6y)e^{-(x^2+y^2)}$;

$f''_{xy} = f''_{yx} = (-2x)e^{-(x^2+y^2)} + (1-2x(x+y))e^{-(x^2+y^2)}(-2y) =$
$(4x^2y+4xy^2-2x-2y)e^{-(x^2+y^2)}$.

$\mathcal{H}_f(P_{12}) = \begin{bmatrix} \mp 3e^{-1/2} & \mp e^{-1/2} \\ \mp e^{-1/2} & \mp 3e^{-1/2} \end{bmatrix}$, $\mathcal{H}_f^1 = \mp 3e^{-1/2}$,

$\mathcal{H}_f^2 = \begin{vmatrix} \mp 3e^{-1/2} & \mp e^{-1/2} \\ \mp e^{-1/2} & \mp 3e^{-1/2} \end{vmatrix} = 9e^{-1} - e^{-1} = 8e^{-1} > 0$. $P_1 = \left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ is a point of maximum with $f\left(\dfrac{1}{2}, \dfrac{1}{2}\right) = e^{-1/2}$ and $P_2 = \left(-\dfrac{1}{2}, -\dfrac{1}{2}\right)$ is a point of minimum with $f\left(-\dfrac{1}{2}, -\dfrac{1}{2}\right) = -e^{-1/2}$.

II M 4) Given the quadratic function $q: \mathbb{R}^3 \to \mathbb{R}$ with associated matrix
$\mathbb{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & k \end{bmatrix}$, where k is real parameter; study, varing the parameter k, the sign of q.

The three principal minors of order one for matrix $\mathbb{Q}$ are $\mathbb{Q}_1^1 = 1$, $\mathbb{Q}_2^1 = 1$, $\mathbb{Q}_3^1 = k$, the three principal minors of order two for matrix $\mathbb{Q}$ are $\mathbb{Q}_{12}^2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$,
$\mathbb{Q}_{13}^2 = \begin{vmatrix} 1 & 0 \\ 0 & k \end{vmatrix} = k$, $\mathbb{Q}_{23}^2 = \begin{vmatrix} 1 & 1 \\ 1 & k \end{vmatrix} = k-1$, and finally the unique principal minor of order three for matrix $\mathbb{Q}$ is $\mathbb{Q}^3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & k \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & k \end{vmatrix} = k-1$. We note that if $k > 1$ all the principal minors of the matrix $\mathbb{Q}$ are positive, thus q is positive defined, if $k < 1$, one of the principal minors of even order for matrix $\mathbb{Q}$ is negative, in this case q is undefined; in the last case ($k = 1$) all the principal minors of the matrix $\mathbb{Q}$ are not negative and at least one principal minor is null, thus q is positive semi-defined.