3. Solow's' exogenous growth

3.1. Introduction

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3.2. The per capita production function

According to neoclassical theory, economic growth is determined on the supply side as a result of the accumulation of productive factors, as well as technical progress. As well know, in fact, marginalist theory considers Say's Law valid, so there are never market problems for the social product. In particular, any amount of savings will find employment in the form of productive investment, just as the flexibility of real wages ensures full employment. So far, however, we are still examining investment from the point of view of demand, as a component of aggregate demand, but not from the point of view of capital accumulation, i.e. as an increase in production capacity. This is dealt with by the growth theory of which in this chapter we consider the marginalist version.

As well-know, for neoclassical economists the economy tends, in the absence of market rigidities, to full employment. These economists are therefore not so much interested in the growth of the aggregate output in order to assure full employment (which is guaranteed by markets flexibility), as in the growth of per capita income as an index of welfare. With structural unemployment, we would be primarily interested in the growth of the aggregate GDP having reabsorption of unemployment in mind. With only frictional unemployment, we feel more interested in the increase of individual well-being identified in per capita GDP. This is of course a crude measure of welfare that does not take into account the many negative externalities of economic development such as pollution, the deterioration of traditional social relations and so on.

Let's start with the standard production function $Y = A\,F(K, N)$. Y is gross output (or GDP), K and N are the two production factors, capital and labour respectively, and A represents technical progress. The production function immediately suggests that increases of K and N determine a growth of Y. Technical progress, an increase in A, has also an autonomous positive effect on the output size. This remained the marginalist intuition about economic growth until in 1956 the American economist Robert Solow gave it a more rigorous shape. His model is still today the workhorse of neoclassical growth theory.

Since we're interested in per capita quantities let us define $y = \frac{Y}{N}$ as the product per worker, and $k = \frac{K}{N}$ the per worker capital stock. On this basis let us write the *per capita function of production*.

First, we divide each term of the aggregate production function by the number of workers: .

$$\frac{Y}{N} = AF(\frac{K}{N}, \frac{N}{N}) = AF(k,1)$$

This function can then be rewritten as, which is the production function in per capita terms:

$$y = Af(k) \qquad (3.1)$$

It tells us that the per capita product depends on the capital endowment per worker. Its graphic form is illustrated in figure 1. Decreasing marginal yields are at work. Given the labour stock *N*, when the *k* increases, the marginal product ${}^{\Delta y}\!/_{\Delta k}$ (which is also the mathematical slope of the function) is progressively lower.

> If we employ a Cobb-Douglass production function $F(K,L) = AK^{\alpha}L^{1-\alpha}$ we get:
>
> $$y = Y/L = F(K,L)/L = F(K/L, L/L) = A\left(\frac{K}{L}\right)^{\alpha}\left(\frac{L}{L}\right)^{1-\alpha}$$
>
> or
>
> $$y = Ak^{\alpha}.$$
>
> There are constant returns to scale since:
>
> $$F(zK, zl) = A(zK)^{\alpha}(zL)^{1-\alpha} = z^{\alpha+1-\alpha}K^{\alpha}L^{1-\alpha} = zF(K,L)$$
>
> The marginal product of capital (useful later) is:
>
> $$MPK = \frac{\partial Y}{\partial K} = \alpha AK^{\alpha-1}L^{1-\alpha}.$$
>
> "α" is capital's income share since:
>
> $$\alpha = \frac{MPK * K}{Y} = \frac{\alpha AK^{\alpha-1}L^{1-\alpha} * K}{AK^{\alpha}L^{1-\alpha}}.$$
>
> We shall use Cobb-Douglass later, when we introduce the endogenous growth models.

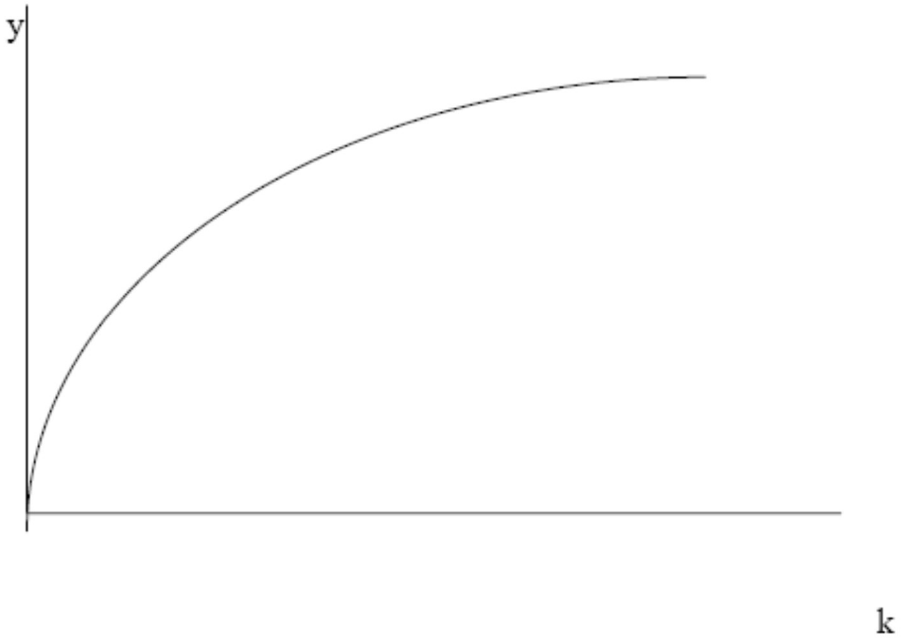

Figure 3.1- Per capita production function

3.3. Stationary states and the fundamental equation of the neoclassical development model

Let's start by studying the so-called "stationary states", long-term equilibriums in which product and capital per worker are constant and technical progress is absent (stationary state does not mean that K, N, and Y do not grow in aggregate). As well know, according to marginalist theory, investments necessary to increase capital per worker require a preliminary act of saving. In addition to providing already active workers with additional equipment, savings also have two other important purposes: to replace capital goods that go out of use due to physical or technical obsolescence, and to equip new workers (young people that enter the labour market, for example) with the same average amount of capital per employee that already equip active workers.

Let's take an example. Suppose our economy consists of a cooperative of 20 people working with 20 computers (one each). A new computer costs 1000€ (the capital stock is therefore worth 20.000€). The cooperative saves 5000€ per year. Suppose also that every year 2 computers, equal to 10% of the total capital goods, go out of use. Then from (gross) saving 2000€ should be set aside to replace the old machines that have gone out of use. Let's also suppose that the cooperative grows by 10% per year and therefore hires 2 young workers. Then another 2000€ of saving should be allocated to equip the two new workers with their own computers. The cooperative still has 1000€ of savings to allocate. These can be used to increase the capital stock per employee, for example by increasing the memory of the PCs in use, precisely from 1000€ each to (1000€ + 1000€/22) ≅ 1045.5€.

We can now analytically translate our parable into the following equation:

$$\Delta kN = sY - \delta K - nK = sY - (\delta + n)K$$

It should be read as follows: sY is the aggregate saving supply; the saving supply can be used to replace capital goods δK that have gone out of use (δ is the share of K that goes out of use in the period considered), or to equip new workers nk (where n is their annual rate of increase: $n = \Delta N / N$).[1] What is left can be used to increase the capital endowment per worker (where Δk is this increase, which we multiply by the number of workers).[2]

Let us now rewrite the expression in per capita terms simply by dividing both sides by N:

$$\frac{\Delta kN}{N} = s\frac{Y}{N} - (\delta + n)\frac{K}{N},$$

that is:

$$\Delta k = sy - (\delta + n)k \qquad (3.2)$$

This is the fundamental equation of Solow's or neoclassical growth model. It tells us that what remains from savings, once replacements are made and new workers are equipped, can be used to increase the per capita capital endowment (all magnitudes expressed in per capita terms).

Equation (3.2) can also be derived in this way:

consider equation (3.3) below which has the obvious meaning that the capital stock can increase if something is left from the saving supply (sY) once we have replaced the worn-out plants (δK):

$$dK / dt = \dot{K} = sY - \delta K \qquad (3.3)$$

Recall the "take logs and then differentiate" rule:

$$k \equiv K / L \Rightarrow \log k = \log K - \log L \Rightarrow \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}.$$

Write equation (3.3) as:

$$\dot{K} / K = sY / K - \delta$$

and recalling that $\frac{\dot{K}}{K} = \frac{\dot{k}}{k} - \frac{\dot{L}}{L}$, we get $\frac{\dot{k}}{k} = \frac{sY/L}{K/L} - n - \delta$ or

$$\dot{k} = sy - (n + \delta)k \qquad \text{(3.2 repeated)}$$

[1] Note that $nK = (K/N)\Delta N = k\Delta N$. This expression makes it clear that nK is the average capital equipment demand k that comes from the new workers that are ΔN.

[2] If we applied the formula to the example we would have 1000€ = 5000€ - 2000€, or more in detail: 45.5€ x 22 = 5000€ - (0.1 x 20.000€) - (0.1 x 20.000€).

Equation (3.2) has a rather straightforward graphic transposition (Fig. 3.2). The right hand side of the equation, Δk, is the difference between the two two expressions on the right side: *sy* and $(\delta + n)k$. Let us examine them. The *sy* (or *sAf(k)*) function has the same graphic shape of fig. 3.1, only a bit scaled down, so to speak, because we take a share of *y* equal to the propensity to save, e.g. if $s = 0.2$, the ordinate is $0.2y$. The $(\delta + n)k$ is a straight line of angular coefficient $(\delta + n)$. The figure indicates Δk as the difference between *sy* and $(\delta + n)k$.

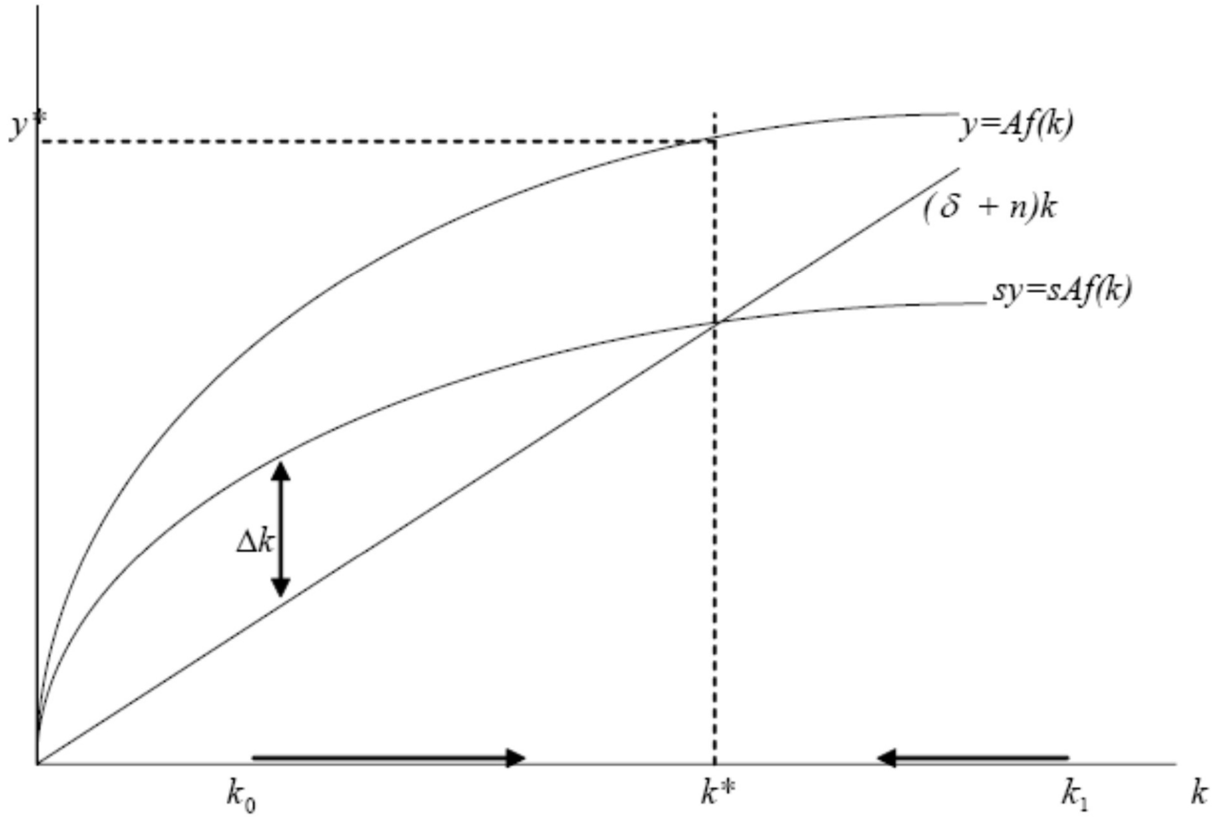

Figure 3.2 – The neoclassical fundamental growth equation

Observe in fig. 3.2 that at the point of intersection between functions *sy* and the $(\delta + n)k$, that is where:

$$sy^* = (\delta + n)k^*,$$

Δk is equal to zero. This is the stationary state (or steady state), i.e. the point at which the saving supply is precisely sufficient to meet the capital demanded for replacements and to equip new workers (all measured per capita terms), but nothing advances to increase the capital endowment per worker. This was instead the case on the left of k* when, in fact, Δk was positive.

Let's pause for a moment on the steady state and make two observations:

a) In order to save notation, let us neglect the replacement rate δ (that is assume that the capital stock does not depreciate) and note that at the stationary state point $nk^* = s^*y$. Since $y/k = (Y/L)/(K/L) = Y/K = 1/v$, we obtain $n = \frac{sy^*}{k^*} = \frac{s}{v^*}$. Hidden behind the Solowian stationary state we find Harrod-Domar warranted growth rate $g^* = s/v^*$! Moreover, this is also equal to the rate of growth of the labour force, so the Solowian growth rate is a full employment rate! The first result does not come as a surprise if we interpret the warranted rate as a saving-investment (or aggregate demand-aggregate supply) equilibrium path. Any equilibrium path must therefore hid the warranted rate, and Solow will not be the first model to follow this rule. In a sense this shows that in equilibrium all models (or theories)[3] are grey, so to speak (see box). What is interesting about models is their respective out-of-equilibrium behaviour. For instance, what it is interesting about Solow's model, as we shall shortly see, is how it converges to the full employment growth rate n.

It is also little noticed that once a Solowian (and for that matter, any growth model) has reached the stationary state, with expectations adjusted to the equilibrium growth rate, we can well argue that in that position both the multiplier and the accelerator are fully and consistently working. More specifically, while out-of-equilibrium investment depends (also) on the interest rate and on a changing capital-intensity of techniques (the variability of the interest rate being itself a manifestation of the disequilibrium), once in the long-run position investment may be regarded as dependent on the fully-adjusted long-run expectations about aggregate demand. The long-run level and growth of investment can then be thought to generate, through the working of the multiplier, the correspondent amount and growth of capacity savings. Using traditional terms, the out-of-equilibrium adjustment regards *capital deepening* and is based on factors' substitution mechanism, while once in equilibrium *capital widening* depends on expected demand (on the accelerator).

b) The second observation is that in the stationary state aggregate magnitudes grow at the rate n, that is, the stationarity only concern the per capita magnitudes k and y. Remember in this regard that $s/v^* = n$ is also the growth rate of the (aggregate) capital stock insofar as: $s/v = \frac{I/K}{K/Y} = I/K = \Delta K/K = g_K$. Since both L and K grow at a rate n, then also aggregate income Y grows at $g^* = n$.

[3] Models are less general than theories.

If our economy has a per-capita capital endowment equal to k^*, the per-capita endowment will not change over time. This is called "steady state" (or stationary equilibrium).

Using the Cobb-Douglas production function $y = Ak^{\alpha}$ and the steady state condition $sy - (\delta + n)k = 0$, we can calculate the value of k^*:

$$s(k^*)^{\alpha} - (\delta + n)k^* = 0$$

or

$$k^* = \left(\frac{sA}{n+\delta}\right)^{1/(1-\alpha)}.$$

Substituting in the production function we can also calculate the steady state level of output per-worker

$$y^* = A(k^*)^{\alpha} = A^{1/(1-\alpha)}\left(\frac{s}{n+\delta}\right)^{\alpha/(1-\alpha)} \qquad (3.3)$$

Let us now show how Solow pretend to prove the stability of the economy around the full employment rate n. We shall see that he will heavily rely upon the marginalist theory artillery.

3.4. Stable, perhaps

Textbook presentations of Solow's stability argument are seriously lacking since they assume that savings automatically leads to investment.. We shall nonetheless move from a standard approach since is more intuitive - but we consider readers capable of taking a second, slightly more complex step. We shall take three steps.

a) As we have seen, at the equilibrium point the economy grows at the warranted rate (this is true for any steady state equilibrium path). Unlike, however, the Harrod-Domar equilibrium, the Solow steady state is a stable equilibrium point, i.e. when the economy is not in equilibrium, it tends towards it. This can be at first approximation perceived by noting that to the left of the equilibrium point of fig. 3.2, for example where $k = k_0$, the function sy lies above the function $(\delta + n)k$, therefore $sy > (\delta + n)k$. This means that on the left of the equilibrium point Δk is positive. In other words, on the left of the equilibrium point per capita savings are more than sufficient for replacements and to equip new workers so that part of them can be destined to increase the capital endowment per worker. If $\Delta k > 0$, it means that k is increasing and therefore its value converges towards k^*, as shown by the right-ward oriented arrow in fig. 3.2. Vice versa on the right of the equilibrium point, for example where $k = k_1$, the function sy lies below the function $\delta + n)k$, so $sy < (\delta + n)k$. Therefore, on the right of the equilibrium point Δk is negative, i.e. per capita savings are not sufficient for replacements and to equip new workers, and therefore a part of the individual capital endowment must be allocated to these two purposes and capital endowment per worker k falls. This means that k is decreasing and its value converges towards k^*, as shown by the left-ward oriented arrow in fig. 3.2

In the terms of our parable, on the left of k^* savings of the cooperative are more than enough to replace the computers that have gone out of use and to equip new members, so there is room to increase the power of the computers provided to each one. On the right of k^* the savings of the cooperative are not enough to replace the computers that have gone out of use and to equip the new members, so we must, so to speak, remove a piece of computers from each member to assemble the computers that are missing for the replacements and for the incoming members; the power (and vakue) of the computers that equip each worker is therefore reduced.

b) Outside the parable, the allocation of excess savings that occurs on the left of k^* is not guided by a benevolent dictator (or economist), as textbooks imply[4]. It is necessary to identify the market mechanisms by which the excess of savings translates into investment. The neoclassical model is here implicitly referring to what happens in the saving-investment market. Presumably, according to marginal theory excess savings lead to a decrease in the "natural" interest rate (i_n) in financial markets. The lower cost of capital leads in turn to a greater convenience for entrepreneurs to adopt more capital-intensive techniques, and thus to an increase of k. Remember also that the slope of $y = f(k)$ is the marginal productivity of capital (Pmk), which entrepreneurs compare with the interest rate, which is the opportunity cost of capital: until $Pmk > i$ - on the left of k^* - entrepreneurs have convenience to increase k, until $Pmk = i_n$. The increase of k determines the increase of $v_a = k/f(k)$. In fact, due to decreasing marginal returns on capital, an increase in the numerator k results in a less than proportional increase in the numerator $f(k)$, and the ratio increases. It should observe that all textbook just identify investment and saving decisions, taking s as the investment rate. This is highly misguiding since it obscure the neoclassical mechanisms on which Solow's model relies upon.

This fundamental aspect of the stability of Solow's model is only noticed (to the best of our knowledge) by Hahn and Matthews (1964, p. 790) in a famous survey of economic growth models:

> In its basic form the neo-classical model depends on the assumption that it is always possible and consistent with equilibrium that investment should be undertaken of an amount equal to full-employment savings. The mechanism that ensures this is as a rule not specified. Most neo-classical writers have, however, had in mind some financial mechanism. In the ideal neo-classical world one may think of there being a certain level of the rate of interest (r) that will lead entrepreneurs, weighing interest cost against

[4] Symmetrically, the fall in the capital-per-worker ratio that takes place on the right of k^* is not guided by a benevolent planner.

> expected profits, to carry out investment equal to full-employment savings. In the absence of risk, etc., the equilibrium rate of interest would equal the rate of profit on investment; otherwise the rate of profit will be higher by the requisite risk premium. As we are at this stage concerned only with the possibility and characteristics of steady growth we may assume that initially the capital stock is that appropriate to steady growth, so that the rate of interest that makes investment equal to full-employment saving in the short period is also the rate of interest required in steady growth.

To sum up, the "financial mechanism" that "neo-classical writers have… had in mind" in the event of an excess savings (left of k^*) is that this would lead to a fall in the interest rate in the financial market. This will in turn induce entrepreneurs to adopt more capital intensive techniques and absorb excess savings. This adjustment will continue until $k = k^*$. At that point the interest rate in the financial market is at its "natural" level at which firms "carry out investment equal to full-employment savings".

c) In the third step we intend to look at the gravitation through the lenses of Harrod-Domar growth equations.

Recall that the capital coefficient at k^* is $v^* = \frac{k^*}{y^*}$ and the long period growth rate is $g^* = s/v^*$. Note also that on the left on the equilibrium point at k_0 the capital coefficient is $v_0 = \frac{k_0}{y_0}$ and, given s (which does not depend on k), we may calculate the actual growth rate at k_0 which is: $g_a = \frac{s}{v_0}$. Note that $v_0 < v^*$ since $k_0/y_0 < k^*/y^*$. Indeed, although both $k_0 < k^*$ and $y_0 < y^*$ given the curvature of the production function k_0 is proportionally lower than y_0 with respect to their respective equilibrium values, as a visual inspection would confirm. We have therefore: $g_a = \frac{s}{v_0} > g^* = \frac{s}{v^*}$, that is, on the left hand side of k^* the actual rate of growth is higher than the long period one. The economy will tend, as we know, to the warranted rate g^*. This means that it is the actual rate g_a that converges to g^*. This is not surprising because in the gravitation process both k and y are increasing but the increase in k is larger than that of y, given the marginal decreasing returns that shape the curvature of the production function (a visual inspection will again confirm this). Therefore, the actual v will rise with the rise of k from k_0 to k^*. The intuition is that during the gravitation process firms are increasing the capital intensity of techniques (K/L) and this also rises the capital coefficient ($v_a = K/Y$). The exercise can be replicated, mutatis mutandis, moving from a disequilibrium point on the right hand side of k^*.

It is important that to appreciate that, e.g., on the left hand side of k^*, during the convergence process, $g_a > g^*$. In other words, outside equilibrium the actual rate of growth is higher or lower than in equilibrium, respectively, on the left and in the right of the long period position. This is one the neoclassical explanation why catching up countries grow faster than developed countries: the idea is the former countries are not yet at their steady state equilibrium and in the catching up phase to reach the secular equilibrium growth is faster. This is not surprising since on the left of k^* both a process of capital deepening (the rise of k) and of capital widening (a rise of K) are occurring, while in the equilibrium point only capital widening is taking place.

Let us now study the effects of changes in the value of two of the parameters, s and n, that govern long-term equilibrium.

3.5. Comparative statics I: changes in the rate of growth of the labour force

To begin with, if the propensity to save varies from s to s', the sy function shifts upwards (Fig. 3). The new equilibrium is characterized by a higher steady-state values of k and y, albeit the rate of growth remains the same. Not surprisingly, an increase in the supply of savings has, in this theory, positive effects by generating an increase in capital and per capita income - although it came as a surprise that the growth rate is unaffected, and this shock is at the origin of endogenous growth theory (chapter 4). Since in the new stationary equilibrium per-capita output is higher, during the transition the rate of growth will be higher than the steady state rate n. We have indeed already noticed that outside equilibrium $g_a \neq g^*$ and that, more specifically, $g_a > g^*$ on the left hand side of k^*. This proves that although there are no long run growth effects of a rise in s, *there are transition growth effects*. The question is then the *length of the transition process*. We shall return on this.

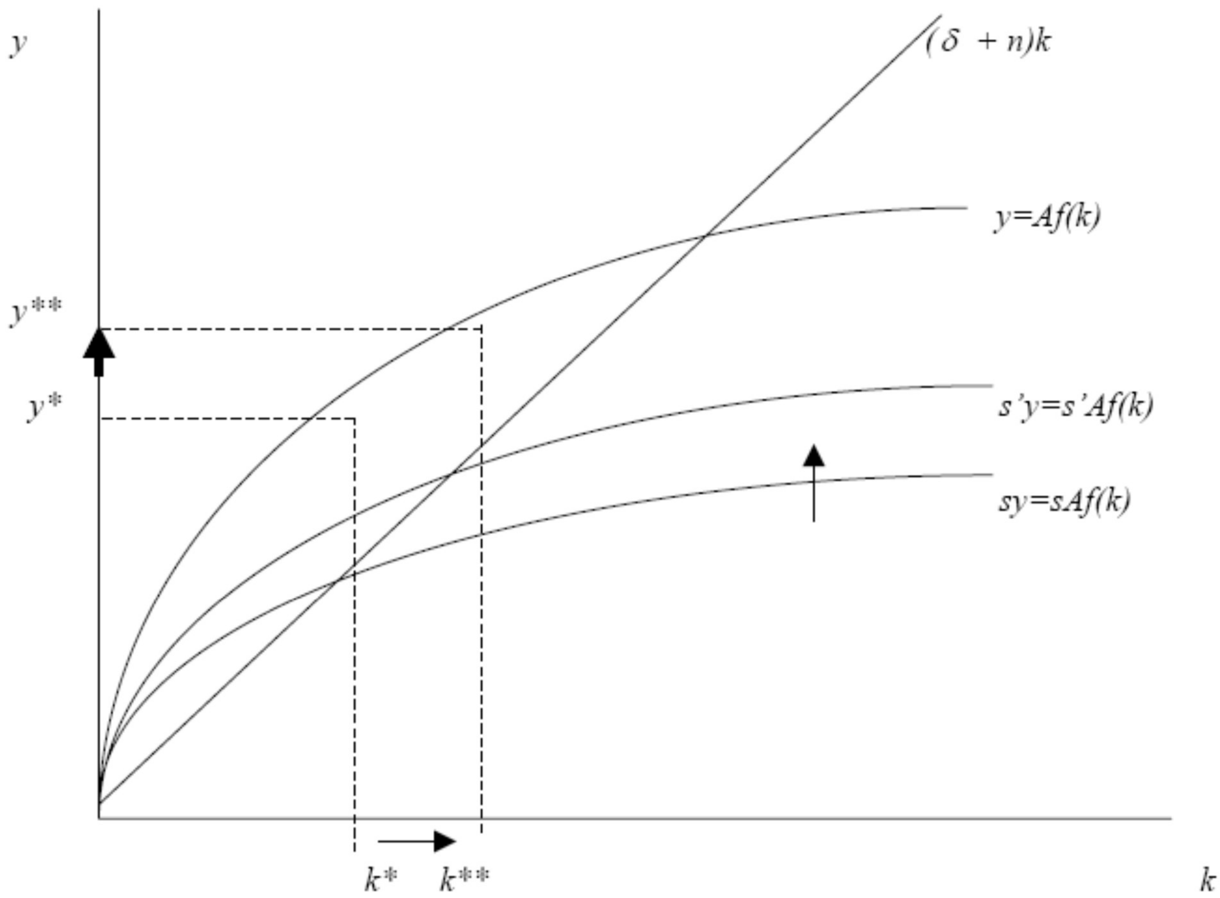

Figure 3.3 – Comparative statics: a variation of the marginal propensity to save

Observation: in the old equilibrium: $g^* = n = s/v_0^*$; while in the new steady state: $g^* = n = s'/v_0^{**}$. This implies that both $s' > s$ and $v_0^{**} > v_0^*$. We already know that when k rises, v follows suit.

The second case is a rise of the rate of growth of the labour force.

If n increases to n', the function $(\delta + n)k$ becomes steeper (Fig. 4); the capital per employee and, consequently, per capita income, decrease. This is not surprising since the saving supply must be used to equip more new workers, and therefore the individual capital endowment must be reduced. Capital is "diluted" (the process opposite to capital deepening).

Observation: in the old equilibrium we had: $g^* = n = s/v_0^*$; while the new one: $g^{**} = n' = s/v_1^{**}$, where $g^* = n < g^{**} = n'$ as $v_0^* > v_1^*$. This is not surprising: when n increases (e.g. due to larger immigration flows), wages decrease due to increased competition and capitalists adopt more labour-intensive (or less capital-intensive techniques), so v decreases.

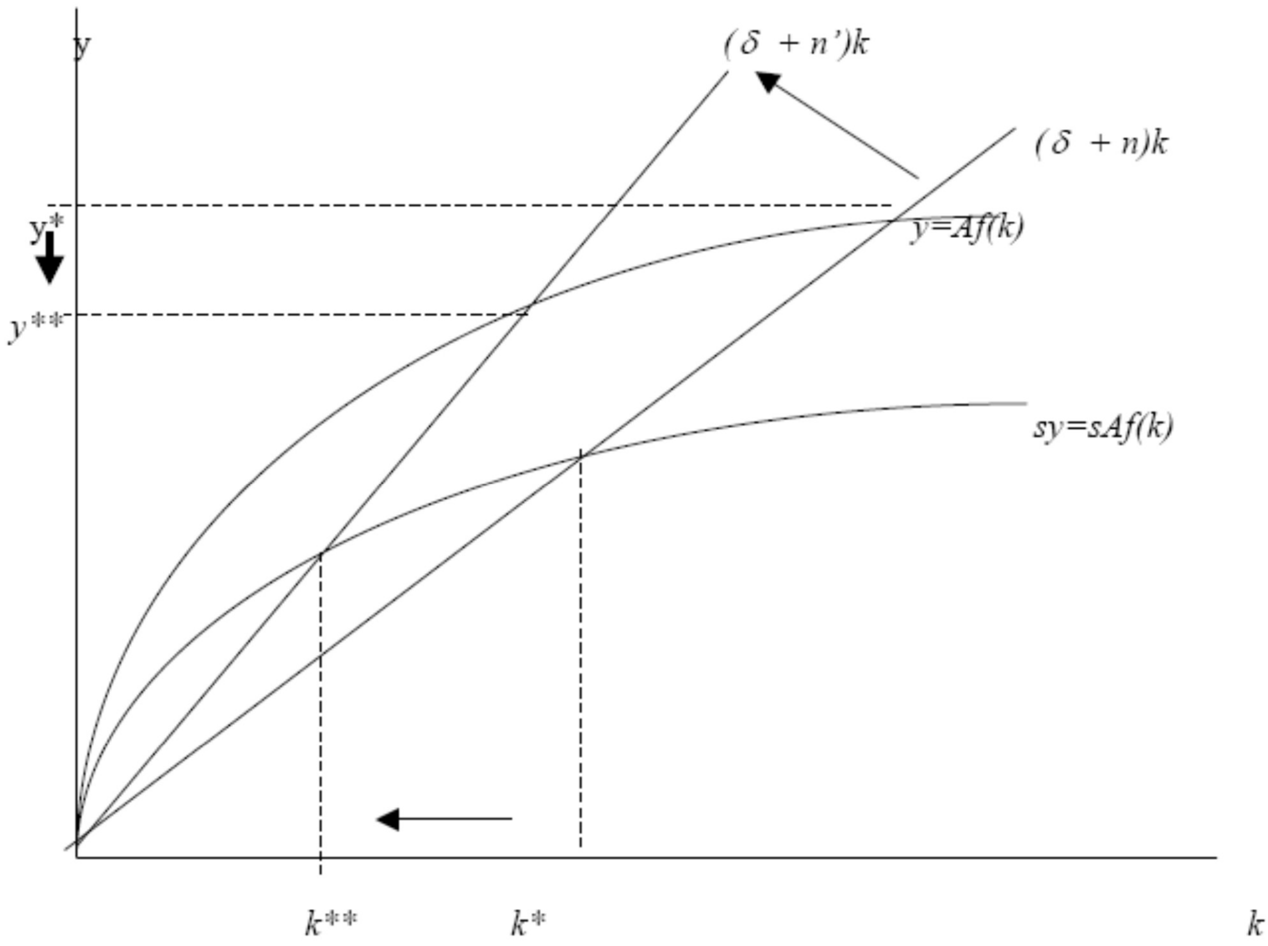

Figure 3.4 - Comparative statics: a variation of the rate of growth of the labour force

3.6. Comparative statics II: technical progress

One may wonder, however, why we should be interested in stationary states, where *k* and *y* are constant. After all, the economic development of the last two hundred years has led to a constant increase in *k and y*. According to the marginalist theory of growth, although an increase of *s* leads to an increase in per capita income, this increase is finite, unless *s* continues to grow, which is not plausible (even if, absurdly, people would save all their income, s could not exceed the value 1). A higher rate of population growth, on the other hand, reduces per capita income. In order to explain the secular growth of the per capita product, we must resort to an element which has been neglected until now: technical progress, indicated by the term *A* in the production function $y = Af(k)$.

Technical progress has many origins. First of all in scientific and technological progress which has given rise to fundamental inventions such as combustion engines, the production and use of electricity, discoveries in chemistry, electronics and so on. All these fundamental inventions give rise to countless other innovations of a more incremental nature. Many minor innovations are also the result of experience in production processes (or so-called learning by doing).[5]

In Solow's model, technical progress is considered exogenous, "like a godsend from heaven", it has often been said. In reality, Solow believed that technical progress was the fruit of human activities, but economists had little systematic to say about its origins. Better then to suppose it exogenous to the model (i.e. not explained by the model). The hypothesis was that $A = A(t)$ (we shall use the expression A_t) increased steadily with the passage of time as suggested by the experience from the industrial revolution onwards, which saw the appearance of an almost constant flow of innovations. Analytically, the rate of technical progress m is defined as: $m = \frac{\Delta A / dt}{A}$. In fig. 3.5 a constant increase over time of A_t manifests itself as a progressive upward shift of the production function. It is easy to observe that technical change leads to a continuous shift of the stationary state equilibrium (a mobile stationary equilibrium, if the oxymoron is forgiven) so that k and y increase in time, as empirical experience indicates.

For reasons that we overlook here, in Solow's model technical progress takes the form "labour augmenting". That is, it increases the power of workers, i.e. it is as if the workforce grew in numbers at the m rate, even if their physical numbers remain constant ($n = 0$). The amount of work in "efficiency units" is defined as $A_t\ L$. In the model with technical progress the economy grows at warranted rate $g^* = n + m$. The capital stock will of course also grow at this combined rate. The n-component of the growth of the capital stock (capital wideming), so to speak, serves to equip new workers with the average capital endowment, in order that y remains constant. The m-component, on the other hand, increases the average capital endowment, so that y increases over time at the m rate capital deepening).[6]

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[5] The literature on technical progress is immense. From the point of view of empirical analysis, the paper favoured by one of the authors is Keith Pavitt (1984). The British economist classifies the different industries and their dominant industrial structure (e.g. small or large enterprises) according to the forms that technical progress takes, e.g. based on science, or learning, or the exchange of information with users and so on. These different forms in turn depend on the combination of certain characteristics that technological knowledge can take on: sources, codification, role of users, appropriability (easily of imitation). The potential of Pavitt's paper has not yet been fully exploited.

[6] In other words, in the equilibrium with technical progress K and Y grow at the rate $n + m$, and k and y at the rate m.

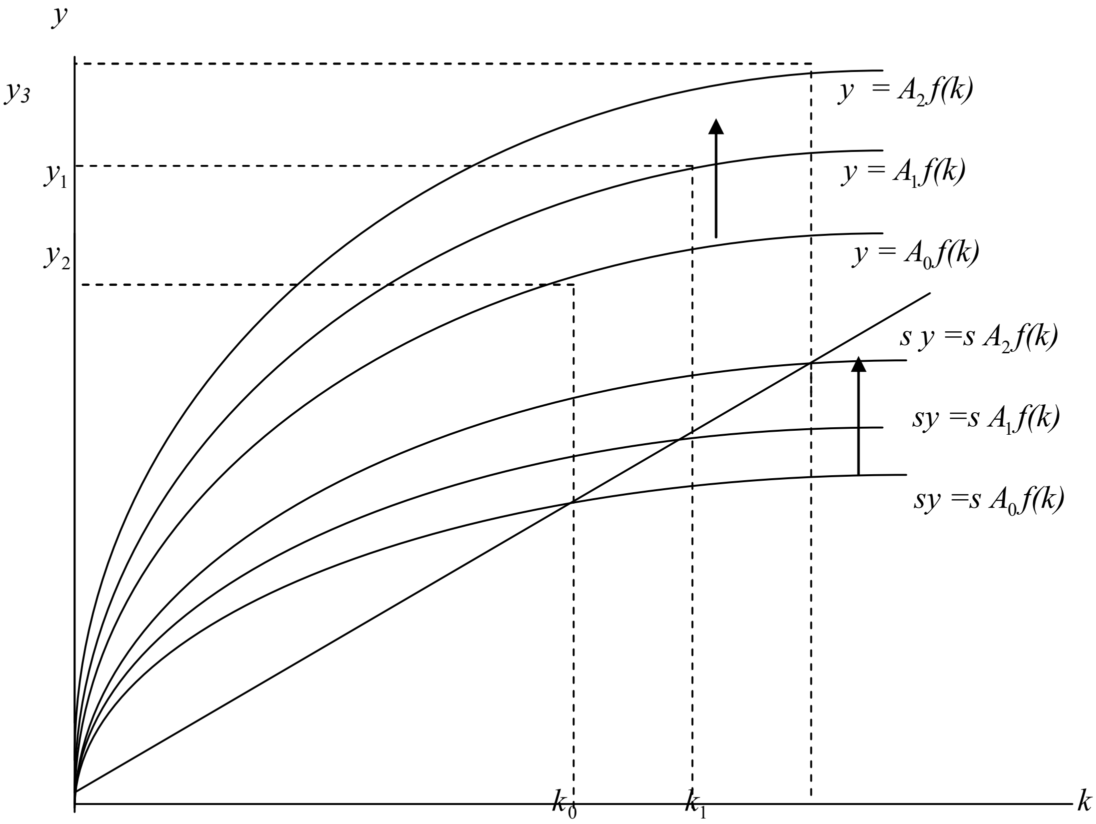
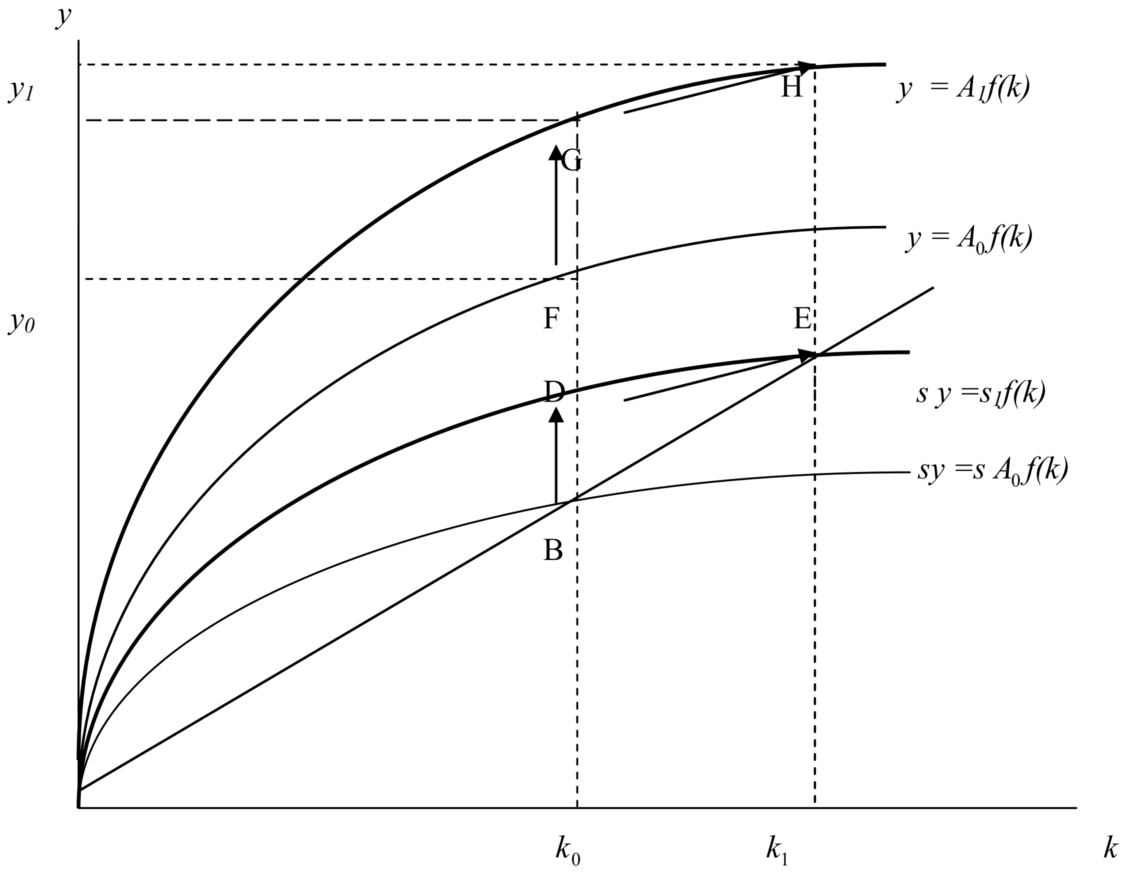


Figure 3.5 - Comparative statics: technical progress I

Figure 3.6 decomposes the increase of output per worker into two passages. To begin with, the shift of the production function determines, given the initial individual capital endowment k_0, a rise of output per worker (the equilibrium temporary moves from F to G). Subsequently, the higher saving supply that derives from the higher income permits a rise of the individual capital endowment and output per worker increases (movement from G to H).



[Correction: the second curve from below is sy = sA$_1$*f(k)]*

Figure 3.6 - Comparative statics: technical progress II

An alternative graphic representation (figure 3.7), similar to fig. 3.2, shows on the abscissae axis the individual capital endowment relative to the amount of work in *A(t)L* efficiency units, i.e. $\widetilde{k} = \frac{K}{A_t L}$.

Income is analogously defined as: $\widetilde{y} = \frac{Y}{A_t L}$, and the production function $\widetilde{y} = f(\widetilde{k})$. The slope of the straight line represents the growth rate of the labour force in efficiency units, i.e. $n + m$.

The fundamental equation of growth can now be written as

$$\Delta\widetilde{k} = s\widetilde{y} - (n+m)\widetilde{k}\,.^{7}$$

At the point of equilibrium $\Delta\widetilde{k} = 0$ and therefore $s\widetilde{y} = (n+m)\widetilde{k}$, that is to say $s\frac{\widetilde{y}}{\widetilde{k}} = n+m$, or

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[7] We are still assuming no depreciation, $\delta = 0$, to save notation.

$$g^* = s/\tilde{v}^* = n + m\,.$$

Note that, as in the case without technical progress, in the steady state we have $g^* = s/v^* = n + m$. Since $s/v^* = S^*/K = I^*/K = \Delta K/K = g_K^*$, where I^* is the investment rate equal to capacity saving at the equilibrium point ($I^* = S^*$), this mean that the aggregate growth rate of the capital stock g_K^* is equal to $n + m$ and so the aggregate rate of output growth is $g_Y^* = g_K^*$. The rates of growth in per capita terms are respectively: $g_k^* = g_K^* - n = m$ and $g_y^* = g_Y^* - n = m$.

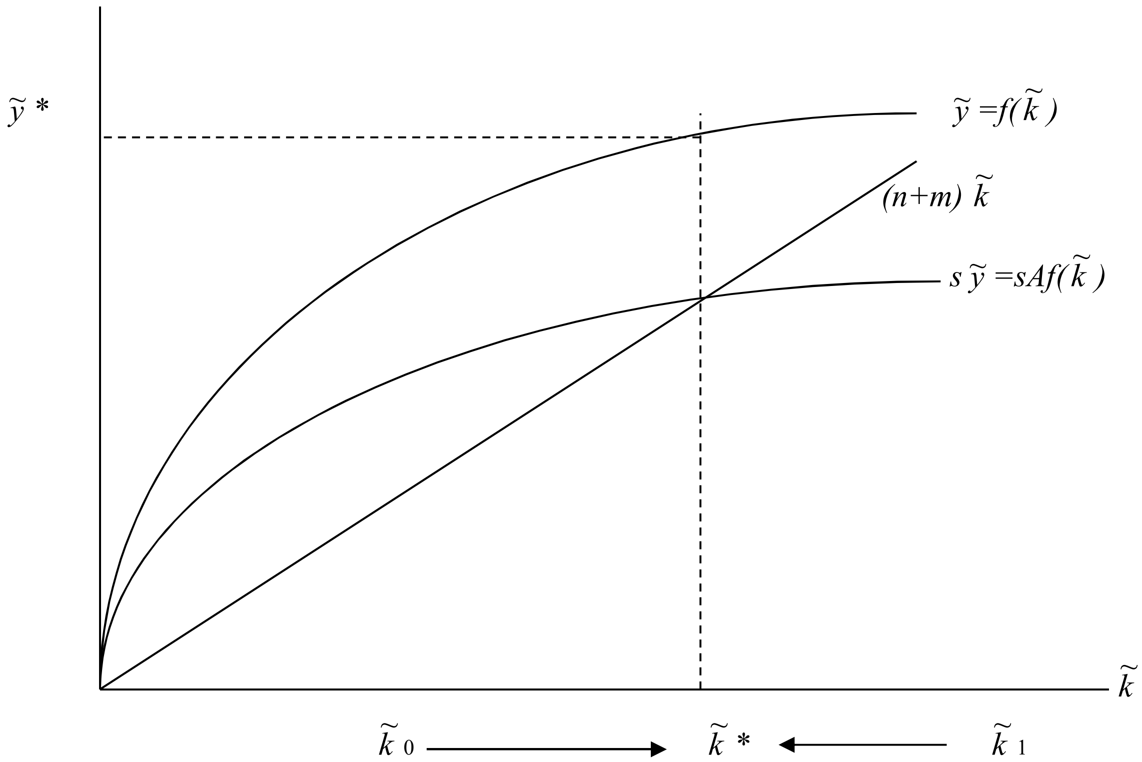


Figura 3.7 - Comparative statics: technical progress III

More analytically, constant exogenous technical change $\dot{A}/A = m$ can be described by the technical progress function (3.4):

$$A = A_0 e^{mt} \qquad (3.4)$$

where the term $m = \dot{A}/A$ represents the constant rate of technical change.

To calculate the steady state values of *k* and *y* let us follow the following steps (C. Jones…).

Step 1

The aggregate and per-capita production functions with technical change can respectively be expressed as:

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha} \qquad (3.5)$$

and

$y = A^{1-\alpha} k^{\alpha}$.

Entered this way technical progress is said to be "labour augmenting".[8]

Define now $\tilde{k} \equiv K / AL$ and $\tilde{y} \equiv Y / AL$ as the pc capital endowment in efficiency units and the pc output in efficiency units, respectively. AL can indeed be considered as the amount of labour employed in the economy augmented by technical progress. The per-capita production function can then be rewritten as $y = \tilde{k}^{\alpha}$. We can now derive again the Solowian fundamental equation of economic growth including technical change this time.

Step 2

Take the log and the derivatives of $\tilde{k} \equiv K / AL$ to obtain:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - m - n \qquad (3.6)$$

Recall that $\dot{K} / K = sY / K - \delta - n$ so to obtain: $\frac{\dot{\tilde{k}}}{k} = \frac{sY / AL}{K / AL} - m - n - \delta$ or:

$\dot{\tilde{k}} = s\tilde{y} - (n + m + \delta)\tilde{k}$.

Step 3

In a steady state equilibrium $\dot{\tilde{k}} = 0$, or $s\tilde{y} = (n + m + \delta)\tilde{k}$ at the point $\tilde{k}*$ (figure 3.6 bis).

Comment

The new equilibrium is not a stationary equilibrium any more, "balanced growth" would be a better definition. At $\tilde{k}*$ aggregate capital is growing at a rate $g = n + m$ (from equation 3.6 with $\dot{\tilde{k}} = 0$), and so aggregate output Y. This growth can be divided in two. There is a <u>capital widening</u> process, since the capital stock grows at the rate of population (or work force) growth n and a <u>capital deepening</u> process, since the pc capital endowment calculated on physical labour is growing at the rate m – just note that in equilibrium $\frac{\dot{K}}{K} = \frac{\dot{A}}{A} + \frac{\dot{L}}{L}$. Carefully observe that in the steady state $\dot{\tilde{k}} = 0$, but $\dot{k} = m$. So even without population growth, k and y are now growing because of exogenous technical progress. Not much else would change compared to the Solow model without technical change (Jones (2013: 39-43; Weil (2009: 239-43))

[8] The other two cases of technical change are "capital augmenting" *F(AK, L)* and neutral *AF(K,L)*. With a Cobb-Douglas production functions the three kind of technical progress are equivalent. For instance, redefining $A = B^{\alpha/(1-\alpha)}$ equation (k) becomes $F(BK, L) = (BK)^{\alpha} (L)^{1-\alpha}$.

If our economy has a per-capita capital endowment equal to k^*, the per-capita endowment will not change over time. This is called "steady state" (or stationary equilibrium).

(*Step 3 cont.)* Using the Cobb-Douglas production function $y = Ak^{\alpha}$ and the steady state condition $\dot{\tilde{k}} = 0$, so that $s\tilde{y} = (n + m + \delta)\tilde{k}$, we can calculate the value of k^*:

$$s(\tilde{k}^*)^{\alpha} - (\delta + n + m)\tilde{k}^* = 0$$

or

$$\tilde{k}^* = \left(\frac{s}{n + m + \delta}\right)^{1/(1-\alpha)}.$$

Substituting in the production function

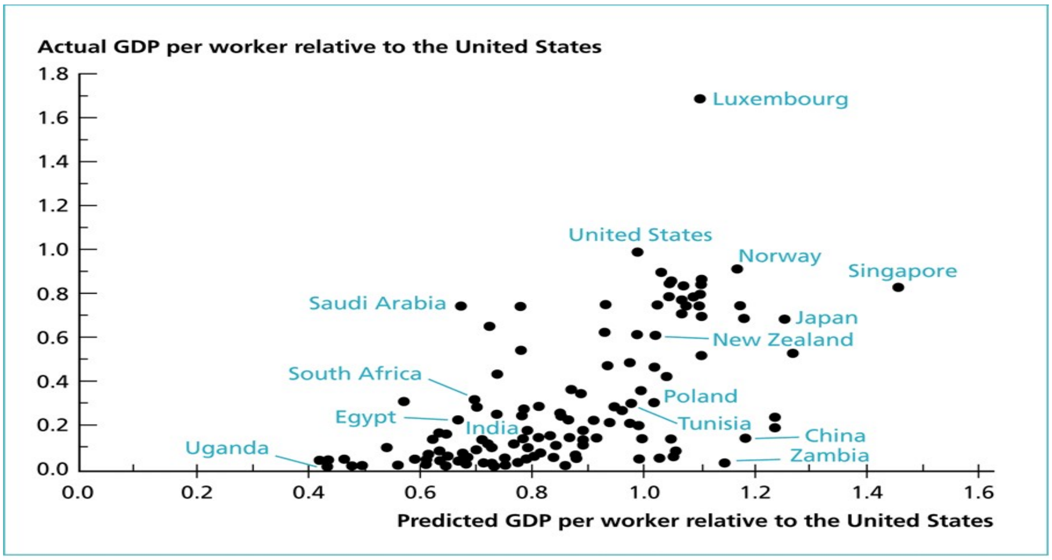
$$\tilde{y}^* = \frac{s}{n + m + \delta}^{\alpha/(1-\alpha)}$$

We can also calculate the steady state level of output per-worker

$$\tilde{y}^* = A(\tilde{k}^*)^{\alpha} = A^{1/(1-\alpha)}\left(\frac{s}{n + m + \delta}\right)^{\alpha/(1-\alpha)} \quad (3.7)$$

3.7. Is Solow's model empirically satisfactory?

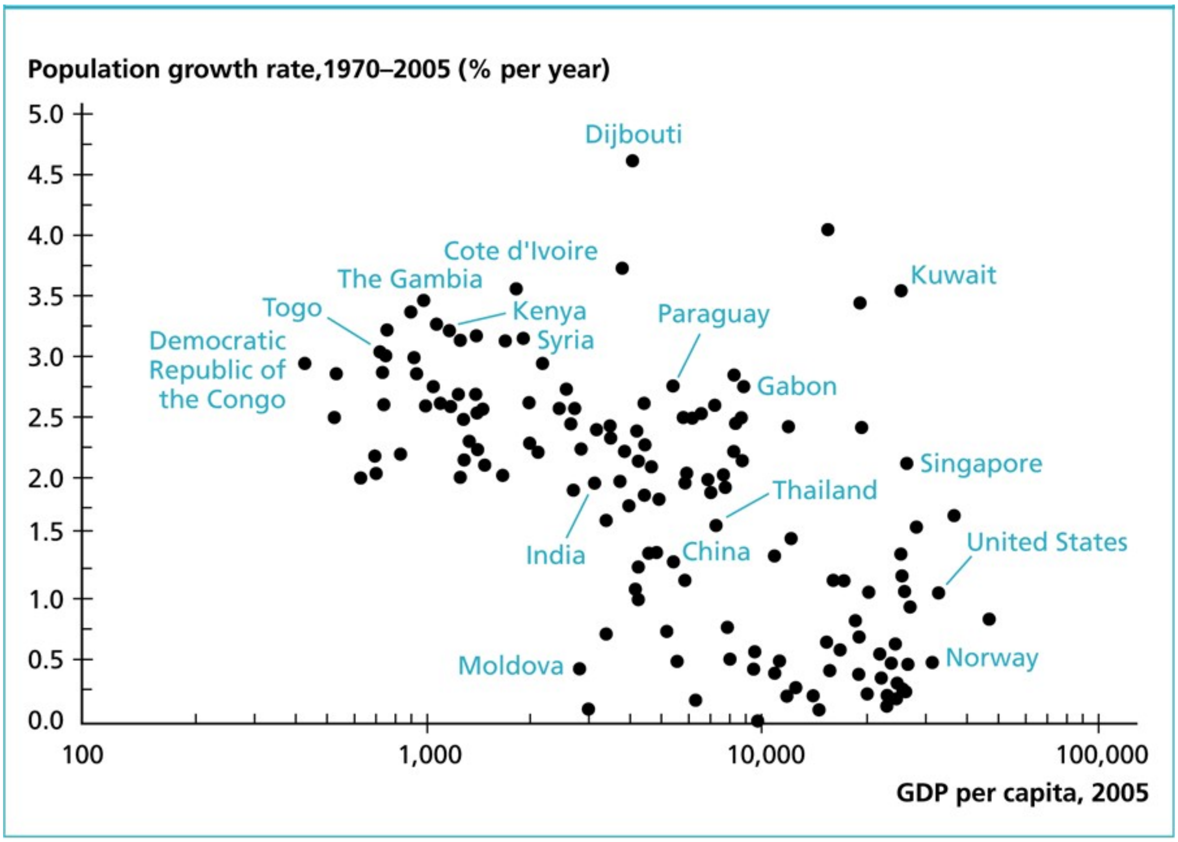
Equation (3.6) shows that, given the share of profits on the income α and the depreciation coefficient δ, il livello of per-capita output positively depends on the saving propensity s (which in steady state is equal to the investment rate, the ratio between I and Y) and negatively on the rate of growth of the labour force. Weil (2009) estimates the predictive power of this result with regard to the actual differences in pc income among countries. Supposing equal the value of the other variables (including population growth rates which are supposed zero, while α is taken at its standard value of 1/3 and δ = 0.05), figure 3.8 shows the actual and the predicted values of the pc GDP relatively to the U.S. of various countries that have different s (= I/K) If the predictive power were perfect, all values would be aligned along the 45° line. As it can seen, countries with a higher investment ratios tend to be wealthier, but the model tend to underestimate the differences (poor countries are predicted richer than they actually are) (ibid: 63-66).



Sources: Calculations based on Heston et al. (2006) and World Bank (2007a).

Figure 3.8

Isolating population growth as the explanatory variable – supposing all the other variables equal – Weil (ibid: 92) estimates a limited explanatory value of the model: a country with zero population growth would have income per worker 34% higher than one with 4% population growth. This would underestimate the real differences in pc income that are shown in figure (3.9):



Source: Heston et.al. (2006), World Bank (2007a).

Figure 3.9

In defence of the explanatory value of Solow's model Weil points out two elements on which we shall return:

(a) if we use a value of $\alpha = 2/3$ in the estimations, then the country with zero population growth would be 3.24 richer than the one with 0.04 growth (ibid: 98). We shall later specify the meaning of this assumption.

(b) particularly developing countries might still have to reach their steady state path, and in the catching up transition they might be investing a large share of output without having reached the corresponding steady state pc output level. (ibid: 66). [From an heterodox point of view we are not surprised that countries in a catching up process invest a large share of output while they still show a low per-capita income].

Weil conclude that while each variable considered in isolation has a limited explanatory value, if we add their explicatory power up, Solow's model does not disfeatures.

3.8. Differences in growth rate and the transition to the steady state

Factor (b) can also help to amend (from a neoclassical point of view, do not forget) a serious gap in this model: its inability to explain the differences of growth rates among countries or, at least, it refers these differences to the exogenous rates of population growth and technical change. In actual, if countries are at different stages of the transition to the steady state, they may show different growth rates among themselves and with countries that have already reached their steady state. Let us investigate the analytics of the transition (Weil: 80-2; Jones 2013: 44).

We know from the above that $\Delta k = sAk^{\alpha} - (\delta + n)k$. Dividing both side by *k*, we obtain the growth rate of the pc capital endowment:

$$\hat{k} = \frac{\Delta k}{k} = sAk^{\alpha - 1} - (\delta + n)$$

Because α is less than one, as *k* rises, the growth rate of *k* gradually declines.

Figure (9) shows the two terms on the right-hand side of this equation. If:

$$sAk^{\alpha-1} > \delta + n --> \hat{k} > 0$$

$$sAk^{\alpha-1} < \delta + n --> \hat{k} < 0$$

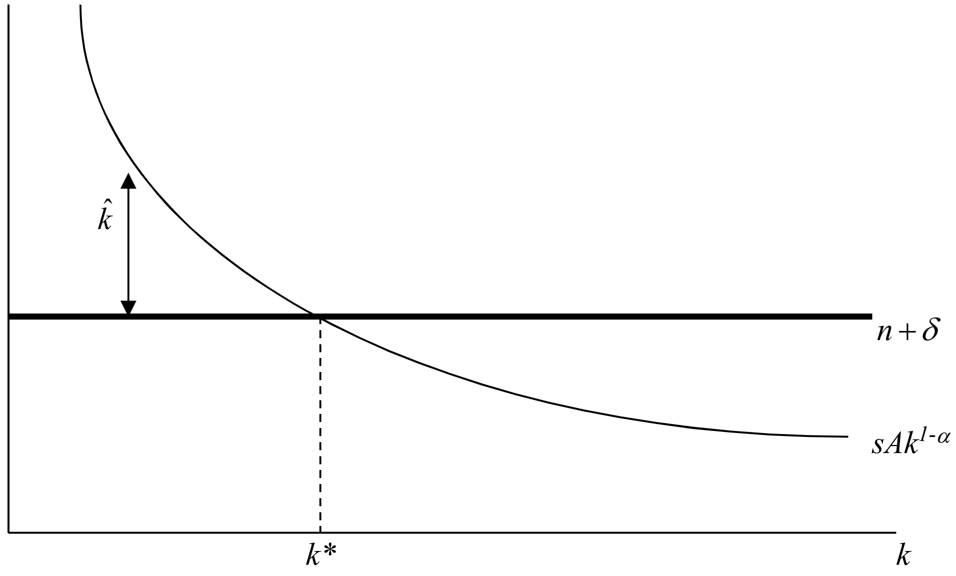


Figure 3.10

Figure (3.10) is an alternative way to look at the Solowian stationary equilibrium. In correspondence to k*, indeed, $sA(k^*)^{\alpha-1} = \delta + n --> \hat{k} = 0$. The figure suggests that the speed of convergence (and therefore the growth rate) is proportional to the distance between the two curves: higher then more fare away is the economy from its steady state equilibrium. In particular, the further an economy is below its steady-state value of *k*, the faster it grows.
(See also the review of the empirical results in Cesaratto (2010), Endogenous growth theory twenty years on: a critical assessment, *Bulletin of Political Economy*, vol.4, n.1, working paper version *Quaderni del Dipartimento di Economia politica*, Università di Siena, n.559)

3.9. Poverty trap

It is natural to believe that ordinary people in poorer countries will have a lower marginal propensity to save than ordinary people in richer countries. Suppose two countries, 1 and 2, in which (see Weil 2009, p. 73) $s = s_1$ if y < y* and s = s_2 if $y > y^*$, respectively, *with* $s_1 < s_2$. Figure (8) shows what happens. It can be seen that the *sf(k)* function jumps in correspondence to the level of income *y* [=f(k*)*]. Country 1 is trapped at equilibrium k_1^{ss} (“ss” stays for steady state) with pc output equal to y_1^{ss}, while country 1 can reach equilibrium k_2^{ss} with a higher pc output y_2^{ss}.

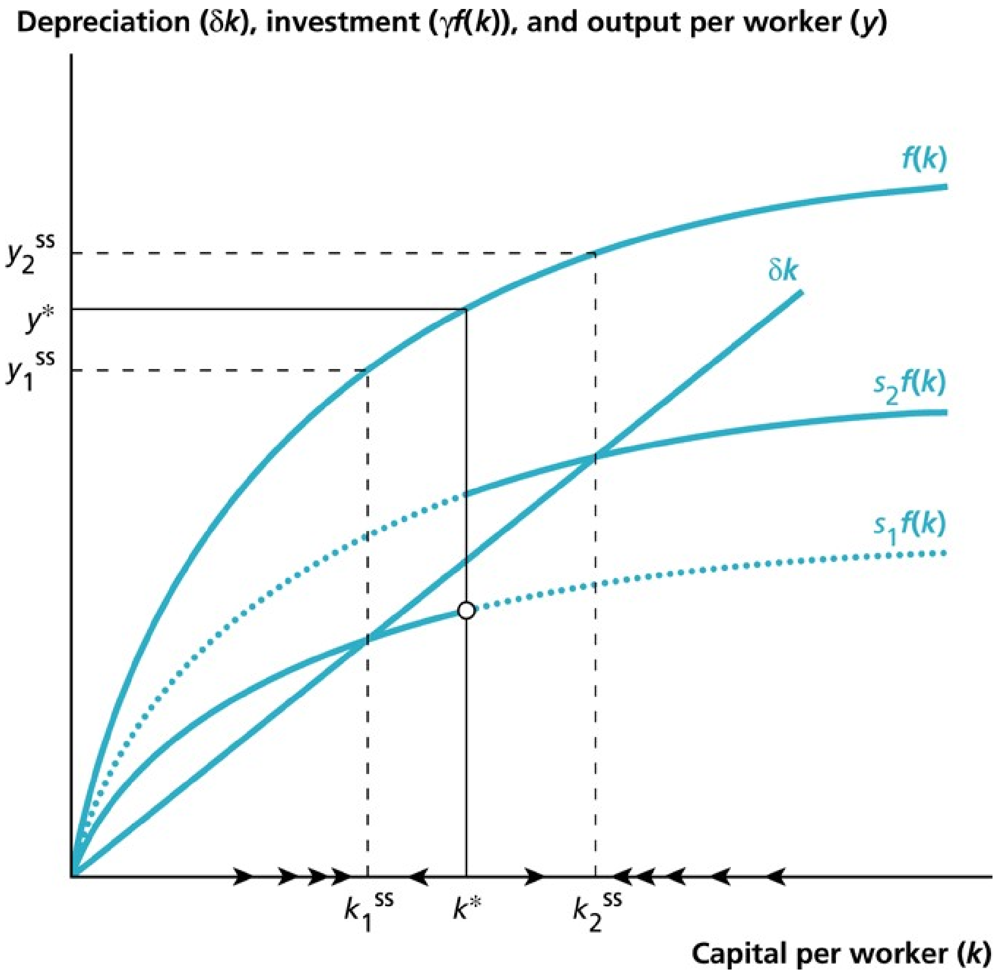


Figure 8

Multiple steady state equilibriums are therefore possible. With little external help to reach a pc capital endowment higher than k^*, country 1 could reach propensity to save s_2 and a better equilibrium. The poverty trap story has thus been used to justify the recourse to foreign official or private capital (saving) to sustain a saving (investment) rate higher than that permitted by domestic saving. This was sometime called the "big push" theory, in the sense that an external push was required for the economy to overcome (in our example) the capital endowment k^* and gravitate towards more satisfactory equilibriums.

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